

ON REAL ANTI-BICANONICAL CURVES WITH ONE DOUBLE POINT ON THE 4-TH REAL HIRZEBRUCH SURFACE

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ABSTRACT. We try to classify real curves with one nondegenerate double point in $|-2K_{\mathbb{F}_4}|$ on the 4-th real Hirzebruch surface $\mathbb{R}\mathbb{F}_4$ (Theorem 6). They are closely related to real 2-elementary K3 surfaces of type $((3, 1, 1), -1)$. We also consider non-increasing simplest degenerations of real nonsingular curves in $|-2K_{\mathbb{F}_4}|$ on $\mathbb{R}\mathbb{F}_4$ (Theorems 14 and 15).

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1. INTRODUCTION

Let \mathbb{F}_m ($m \geq 0$) be the m -th Hirzebruch surface, i.e., the ruled surface over \mathbb{P}^1 having an exceptional section s with $s^2 = -m$. Real structures, i.e., anti-holomorphic involutions, on Hirzebruch surfaces are as follows (See [2]):

$\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ has **4** real structures.

\mathbb{F}_m (m : odd) has a **unique** real structure.

\mathbb{F}_m ($m \geq 2$, even) has **2** real structures.

By a **real** m -th Hirzebruch surface (\mathbb{F}_m, θ) we mean a Hirzebruch surface \mathbb{F}_m equipped with an anti-holomorphic involution θ . We say a complex curve A on (\mathbb{F}_m, θ) is **real** if $\theta(A) = A$, namely, θ can be restricted to A .

Problem 1. *Classify nonsingular real curves in $|-2K_{\mathbb{F}_m}|$ on a real Hirzebruch surface (\mathbb{F}_m, θ) .*

The isotopy types of nonsingular real curves in $|-2K_{\mathbb{F}_m}|$ on the real m -th Hirzebruch surface \mathbb{F}_m were classified by Nikulin and the author for $m = 0, 1, 4$ ([9]), and for $m = 2, 3$ ([10]). Actually, these articles investigate real 2-elementary K3 surfaces, i.e., real K3 surfaces with non-symplectic holomorphic involutions. If the fixed point set of the non-symplectic holomorphic involution is not empty, then it is a nonsingular anti-bi-canonical curve ([7]). Especially, in the cases treated in [9], Hirzebruch surfaces \mathbb{F}_m ($m = 0, 1, 4$) appear as the quotient spaces of the holomorphic involutions, and in [10], \mathbb{F}_m ($m = 2, 3$) appear.

Problem 2. *Classify real curves with one (nondegenerate) double point in $|-2K_{\mathbb{F}_m}|$ on a real Hirzebruch surface (\mathbb{F}_m, θ) .*

In this paper we consider the case $m = 4$. Real curves with one double point in $|-2K_{\mathbb{F}_4}|$ on real \mathbb{F}_4 are related to real 2-elementary K3 surfaces of type $((3, 1, 1), -1)$. In the next section we state the definition of the invariant $((3, 1, 1), -1)$.

2. REAL 2-ELEMENTARY K3 SURFACES

2.1. Real 2-elementary K3 surfaces.

Definition 1 (Real 2-elementary K3 surface). *A triple (X, τ, φ) is called a **real K3 surface with non-symplectic (holomorphic) involution** or **real 2-elementary K3 surface** if*

- (1) (X, τ) is a K3 surface X with a **non-symplectic holomorphic involution** τ , i.e., “2-elementary K3 surface” ([7]).
- (2) φ is an anti-holomorphic involution on X .
- (3) $\varphi \circ \tau = \tau \circ \varphi$ \square

For a 2-elementary K3 surface (X, τ) , we denote by

$$H_{2+}(X, \mathbb{Z})$$

the fixed part of $\tau_* : H_2(X, \mathbb{Z}) \rightarrow H_2(X, \mathbb{Z})$.

We fix an even unimodular lattice \mathbb{L}_{K3} of signature $(3, 19)$. The isometry class of such lattices is unique (**the K3 lattice**).

Let $\alpha : H_2(X, \mathbb{Z}) \rightarrow \mathbb{L}_{K3}$ be an isometry (so-called a marking). If we set

$$S := \alpha(H_{2+}(X, \mathbb{Z})),$$

then S is a primitive hyperbolic 2-elementary sublattice of the K3 lattice \mathbb{L}_{K3} . We say S is the **type** of a 2-elementary K3 surface (X, τ) ([7]).

Now let S be a primitive hyperbolic 2-elementary sublattice of the K3 lattice \mathbb{L}_{K3} .

Remark 1 ([1], [6]). *It is known that there exists a unique primitive embedding $S \rightarrow \mathbb{L}_{K3}$ up to the automorphisms of \mathbb{L}_{K3} . That is, if S, S' be primitive hyperbolic 2-elementary sublattices of the K3 lattice \mathbb{L}_{K3} and S and S' are isometric, then there is an automorphism f of \mathbb{L}_{K3} such that $f(S') = S$. \square*

Remark 2. *Let (X, τ) and (X', τ') be two 2-elementary K3 surfaces. Suppose that $H_{2+}(X, \mathbb{Z})$ and $H_{2+}(X', \mathbb{Z})$ are isometric. Let $\alpha : H_2(X, \mathbb{Z}) \rightarrow \mathbb{L}_{K3}$ be an isometry and set $S := \alpha(H_{2+}(X, \mathbb{Z}))$. Let $\alpha' : H_2(X', \mathbb{Z}) \rightarrow \mathbb{L}_{K3}$ be an isometry and set $S' := \alpha'(H_{2+}(X', \mathbb{Z}))$. Since $H_{2+}(X, \mathbb{Z})$ and $H_{2+}(X', \mathbb{Z})$ are isometric, S and S' are also isometric. By the preceding remark, there is an automorphism f of \mathbb{L}_{K3} such that $f(S') = S$. Let us consider the new isometry $f \circ \alpha'$. Then*

$(f \circ \alpha')(H_{2+}(X', \mathbb{Z})) = f(S') = S$. If we set $\alpha'' := f \circ \alpha'$, then it is concluded that $\alpha'' : H_2(X', \mathbb{Z}) \rightarrow \mathbb{L}_{K3}$ is an isometry (a marking) with the property that $\alpha''(H_{2+}(X', \mathbb{Z})) = S$. \square

Definition 2 (Genus invariants). We set $r(S) := \text{rank } S$. The non-negative integer $a(S)$ is defined by

$$S^*/S \cong (\mathbb{Z}/2\mathbb{Z})^{a(S)}.$$

We define

$$\delta(S) := \begin{cases} 0 & \text{if } z \cdot \tau(z) \equiv 0 \pmod{2} \quad (\forall z \in \mathbb{L}_{K3}) \\ 1 & \text{otherwise,} \end{cases}$$

where we temporary set $\tau : \mathbb{L}_{K3} \rightarrow \mathbb{L}_{K3}$ to be the involution whose fixed part is S . (Remark that $\delta(S) = 0$ if and only if $(x^*)^2 \in \mathbb{Z}$ for any $x^* \in S^*$.) \square

Remark 3 ([7]). The triplet $(r(S), a(S), \delta(S))$ determines the isometry class of S . \square

Let (X, τ, φ) be a **real** 2-elementary K3 surface, namely, (X, τ) is a 2-elementary K3 surface, i.e., a K3 surface X with a non-symplectic holomorphic involution τ , and φ is an anti-holomorphic involution on X with $\varphi \circ \tau = \tau \circ \varphi$, and (X', τ', φ') also be a real 2-elementary K3 surface.

If the fixed lattices $H_{2+}(X, \mathbb{Z})$ and $H_{2+}(X', \mathbb{Z})$ are isometric, then by the above remark, there exist markings α, α' such that $\alpha(H_{2+}(X, \mathbb{Z})) = \alpha'(H_{2+}(X', \mathbb{Z})) =: S$.

Now let θ be an involution of S . We don't know whether we can moreover take α, α' such that $\alpha \circ \varphi_* = \theta \circ \alpha$ on $H_{2+}(X, \mathbb{Z})$ and $\alpha' \circ \varphi'_* = \theta \circ \alpha'$ on $H_{2+}(X', \mathbb{Z})$ for the same θ .

Definition 3 (the action of φ on $H_{2+}(X, \mathbb{Z})$). Let (X, τ, φ) be a real 2-elementary K3 surface. If there exists an isometry (a marking)

$$\alpha : H_2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$$

such that $\alpha(H_{2+}(X, \mathbb{Z})) = S$ and $\alpha \circ \varphi_* = \theta \circ \alpha$ on $H_{2+}(X, \mathbb{Z})$ (the fixed part of τ_*).

$$\begin{array}{ccc} & \alpha & \\ H_{2+}(X, \mathbb{Z}) & \rightarrow & S \\ \varphi_* \downarrow & & \downarrow \theta \\ H_{2+}(X, \mathbb{Z}) & \rightarrow & S \\ & \alpha, & \end{array}$$

then we call (X, τ, φ) a 2-elementary K3 surface of **type** (S, θ) . \square

Definition 4 (marked real 2-elementary K3 surfaces). A pair

$$((X, \tau, \varphi), \alpha)$$

of a real 2-elementary K3 surface (X, τ, φ) of type (S, θ) and an isometry (a marking)

$$\alpha : H_2(X, \mathbb{Z}) \cong \mathbb{L}_{K3}$$

such that $\alpha(H_{2+}(X, \mathbb{Z})) = S$ and $\alpha \circ \varphi_* = \theta \circ \alpha$ on $H_{2+}(X, \mathbb{Z})$ is called a **marked real 2-elementary K3 surface of type** (S, θ) . \square

2.2. Involutions of the K3 lattice \mathbb{L}_{K3} of type (S, θ) .

Definition 5 (Involutions ψ of \mathbb{L}_{K3} of type (S, θ)). Let S be a hyperbolic 2-elementary sublattice of \mathbb{L}_{K3} , $\theta : S \rightarrow S$ be an involution of lattice S , and $\psi : \mathbb{L}_{K3} \rightarrow \mathbb{L}_{K3}$ be an involution of the lattice \mathbb{L}_{K3} such that $\psi(S) = S$, $\psi|_S = \theta$.

$$\begin{array}{ccc} S & \subset & \mathbb{L}_{K3} \\ \theta \downarrow & & \downarrow \psi \\ S & \subset & \mathbb{L}_{K3} \end{array}$$

We say such a pair (\mathbb{L}_{K3}, ψ) (or ψ) an **involution of \mathbb{L}_{K3} of type (S, θ)** . \square

Remark 4. Let $((X, \tau, \varphi,) \alpha)$ be a marked real 2-elementary K3 surface of type (S, θ) . If we set

$$\psi := \alpha \circ \varphi_* \circ \alpha^{-1} : \mathbb{L}_{K3} \rightarrow \mathbb{L}_{K3},$$

then we have $\psi(S) = \alpha \circ \varphi_* \circ \alpha^{-1}(\alpha(H_{2+}(X, \mathbb{Z}))) = \alpha \circ \varphi_*(H_{2+}(X, \mathbb{Z})) = \theta \circ \alpha(H_{2+}(X, \mathbb{Z})) = \theta(S) = S$. For every $x \in S$, $\psi(x) = \alpha \circ \varphi_* \circ \alpha^{-1}(x) = \theta \circ \alpha \circ \alpha^{-1}(x) = \theta(x)$ because $\alpha^{-1}(x) \in H_{2+}(X, \mathbb{Z})$. Hence, (\mathbb{L}_{K3}, ψ) is an involution of \mathbb{L}_{K3} of type (S, θ) . \square

Definition 6 (the associated involution). We call the involution $\psi := \alpha \circ \varphi_* \circ \alpha^{-1}$ of \mathbb{L}_{K3} of type (S, θ) the associated (integral) involution of \mathbb{L}_{K3} with a marked real 2-elementary K3 surface $((X, \tau, \varphi,) \alpha)$ of type (S, θ) .

$$\begin{array}{ccc} & \alpha & \\ H_2(X, \mathbb{Z}) & \xrightarrow{\quad} & \mathbb{L}_{K3} \\ \varphi_* \downarrow & & \psi \downarrow \\ H_2(X, \mathbb{Z}) & \xrightarrow{\quad} & \mathbb{L}_{K3} \\ & \alpha & \end{array}$$

We now introduce the subgroup “ G ” of $W^{(-4)}(S, L)_{\mathcal{M}}$. It is the real analogy of the group $W^{(-4)}(S, L)_{\mathcal{M}}$. See [9] for the definitions of $W^{(-4)}(S, L)_{\mathcal{M}}$ e.t.c..

Definition 7 (The group G , see [9]). We define G to be the subgroup generated by reflections s_{δ_1} in all elements $\delta_1 \in \Delta(S, L)^{(-4)}$ which are contained in S_+ or S_- ($\Leftrightarrow s_{\delta_1}$ commutes with θ) and such that $s_{\delta_1}(\mathcal{M}) = \mathcal{M}$. \square

Remark 5. For example, if $(S, \theta) = (\langle 2 \rangle \oplus \langle -2 \rangle, -1)$, $((3, 1, 1), -1)$, then we have $G = \{1_S\}$. \square

Definition 8. Let $(\mathbb{L}_{K3}, \psi_1)$ and $(\mathbb{L}_{K3}, \psi_2)$ be two involutions of \mathbb{L}_{K3} of type (S, θ) . An **isometry with respect to the group G** from $(\mathbb{L}_{K3}, \psi_1)$ to $(\mathbb{L}_{K3}, \psi_2)$ means an isometry

$$f : \mathbb{L}_{K3} \rightarrow \mathbb{L}_{K3}$$

such that $\psi_2 \circ f = f \circ \psi_1$, $f(S) = S$, and $f|_S \in G$.

$$\begin{array}{ccc} & f & \\ \mathbb{L}_{K3} & \xrightarrow{\quad} & \mathbb{L}_{K3} \\ \psi_1 \downarrow & & \downarrow \psi_2 \\ \mathbb{L}_{K3} & \xrightarrow{\quad} & \mathbb{L}_{K3} \\ & f & \end{array}$$

Remark that $\theta \circ f|_S = f|_S \circ \theta$ on S , namely, $f|_S$ is an automorphism of (S, θ) .

Definition 9. Two involutions $(\mathbb{L}_{K3}, \psi_1)$ and $(\mathbb{L}_{K3}, \psi_2)$ of type (S, θ) are **isometric with respect to the group G** if there exists an isometry with respect to the group G from $(\mathbb{L}_{K3}, \psi_1)$ to $(\mathbb{L}_{K3}, \psi_2)$. \square

Definition 10. By an **automorphism of an involution (\mathbb{L}_{K3}, ψ) with respect to the group G** of type (S, θ) we mean an isometry with respect to the group G from (\mathbb{L}_{K3}, ψ) to itself. Namely, an isometry $f : \mathbb{L}_{K3} \rightarrow \mathbb{L}_{K3}$ which satisfies that $\psi \circ f = f \circ \psi$, $f(S) = S$ and $f|_S \in G$. \square

Definition 11 (analytically isomorphic with respect to G). Two marked real 2-elementary K3 surfaces $((X, \tau, \varphi), \alpha)$ and $((X', \tau', \varphi'), \alpha')$ of type (S, θ) are **(analytically) isomorphic with respect to the group G** if there exists an analytic isomorphism $f : X \rightarrow X'$ such that $f \circ \tau = \tau' \circ f$, $f \circ \varphi = \varphi' \circ f$ and $\alpha' \circ f_* \circ \alpha^{-1}|_S \in G$. \square

If two marked real 2-elementary K3 surfaces $((X, \tau, \varphi), \alpha)$ and $((X', \tau', \varphi'), \alpha')$ of type (S, θ) are (analytically) isomorphic with respect to the group G , then we have the following commutative

diagrams.

$$\begin{array}{ccccccc}
& & f_* & & & & \\
H_2(X, \mathbb{Z}) & \rightarrow & H_2(X', \mathbb{Z}) & & & & \\
\tau_* \downarrow & & \tau'_* \downarrow & & & & \\
H_2(X, \mathbb{Z}) & \rightarrow & H_2(X', \mathbb{Z}) & & & & \\
& & f_* & & & & \\
\alpha^{-1} & & f_* & & \alpha' & & \\
\mathbb{L}_{K3} & \rightarrow & H_2(X, \mathbb{Z}) & \rightarrow & H_2(X', \mathbb{Z}) & \rightarrow & \mathbb{L}_{K3} \\
\psi \downarrow & & \varphi_* \downarrow & & \varphi'_* \downarrow & & \psi' \downarrow \\
\mathbb{L}_{K3} & \rightarrow & H_2(X, \mathbb{Z}) & \rightarrow & H_2(X', \mathbb{Z}) & \rightarrow & \mathbb{L}_{K3} \\
& & \alpha^{-1} & & f_* & & \alpha'
\end{array}$$

Here we set $\psi := \alpha \circ \varphi_* \circ \alpha^{-1}$ and $\psi' := \alpha' \circ \varphi'_* \circ \alpha'^{-1}$. They are the associated involutions of \mathbb{L}_{K3} .

$$\begin{array}{ccccccc}
& & \alpha^{-1} & & f_* & & \alpha' \\
S & \rightarrow & H_{2+}(X, \mathbb{Z}) & \rightarrow & H_{2+}(X', \mathbb{Z}) & \rightarrow & S \\
\downarrow \theta & & \varphi_* \downarrow & & \varphi'_* \downarrow & & \downarrow \theta \\
S & \rightarrow & H_{2+}(X, \mathbb{Z}) & \rightarrow & H_{2+}(X', \mathbb{Z}) & \rightarrow & S \\
& & \alpha^{-1} & & f_* & & \alpha'
\end{array}$$

Thus we see that

Remark 6. *If two marked real 2-elementary K3 surfaces $((X, \tau, \varphi), \alpha)$ and $((X', \tau', \varphi'), \alpha')$ of type (S, θ) are (analytically) isomorphic with respect to the group G , then their associated involutions ψ and ψ' of \mathbb{L}_{K3} are isometric with respect to the group G . \square*

Let $((X, \tau, \varphi), \alpha)$ and $((X', \tau', \varphi'), \alpha')$ be two marked real 2-elementary K3 surfaces of type (S, θ) . Suppose that their associated involutions ψ and ψ' of \mathbb{L}_{K3} are isometric with respect to the group G . They are not necessarily analytically isomorphic, and there exists an isometry with respect to the group G from (\mathbb{L}_{K3}, ψ) to (\mathbb{L}_{K3}, ψ') , namely, there exists an isometry

$$f : \mathbb{L}_{K3} \rightarrow \mathbb{L}_{K3}$$

such that $\psi' \circ f = f \circ \psi$, $f(S) = S$, and $f|_S \in G$.

$$\begin{array}{ccc}
& f & \\
\mathbb{L}_{K3} & \rightarrow & \mathbb{L}_{K3} \\
\psi \downarrow & & \downarrow \psi' \\
\mathbb{L}_{K3} & \rightarrow & \mathbb{L}_{K3} \\
& f &
\end{array}$$

We have $\alpha' \circ \varphi'_* \circ \alpha'^{-1} \circ f = f \circ \alpha \circ \varphi_* \circ \alpha^{-1}$, and $f^{-1} \circ \alpha' \circ \varphi'_* \circ \alpha'^{-1} \circ f = \alpha \circ \varphi_* \circ \alpha^{-1}$. Hence, $(f^{-1} \circ \alpha') \circ \varphi'_* \circ (f^{-1} \circ \alpha')^{-1} = \alpha \circ \varphi_* \circ \alpha^{-1} = \psi$.

Here, $(f^{-1} \circ \alpha') : \mathbb{L}_{K3} \rightarrow \mathbb{L}_{K3}$ is an isometry and $(f^{-1} \circ \alpha')(H_{2+}(X', \mathbb{Z})) = f^{-1}(S) = S$, and $(f^{-1} \circ \alpha') \circ \varphi'_* = f^{-1} \circ (\alpha' \circ \varphi'_*) = f^{-1} \circ (\theta \circ \alpha')$ on $H_{2+}(X', \mathbb{Z})$.

If we suppose that $G = \{1_S\}$, then $f|_S = 1_S$ and we have $(f^{-1} \circ \alpha') \circ \varphi'_* = \theta \circ \alpha'$ on $H_{2+}(X', \mathbb{Z})$. Hence, $(f^{-1} \circ \alpha') : \mathbb{L}_{K3} \rightarrow \mathbb{L}_{K3}$ is another marking of (X', τ', φ') . If we take this new marking of (X', τ', φ') , then its associated involution is nothing but ψ , which is the same as $((X, \tau, \varphi), \alpha)$. Thus we see that

Remark 7. *Suppose that $G = \{1_S\}$. If two marked real 2-elementary K3 surfaces $((X, \tau, \varphi), \alpha)$ and $((X', \tau', \varphi'), \alpha')$ of type (S, θ) have the isometric associated involutions of \mathbb{L}_{K3} (with respect to the group $G = \{1_S\}$), then, by replacing their markings appropriately if necessary, they have just the same associated involution. \square*

2.3. Periods of marked real 2-elementary K3 surfaces. Let us fix an involution (\mathbb{L}_{K3}, ψ) of type (S, θ) through this subsection. (cf. Remark 7.)

We set

$$\Omega := \{\omega (\neq 0) \in \mathbb{L}_{K3} \otimes \mathbb{C} \mid \omega \cdot \omega = 0, \omega \cdot \bar{\omega} > 0, \omega \cdot S = 0, \psi_{\mathbb{C}}(\omega) = \bar{\omega}\} / \mathbb{R}_*.$$

Let $((X, \tau, \varphi), \alpha)$ be a marked real 2-elementary K3 surface of type (S, θ) satisfying

$$\alpha \circ \varphi_* \circ \alpha^{-1} = \psi,$$

namely, ψ is the associated involution with $((X, \tau, \varphi), \alpha)$.

We denote by $H \subset H_2(X, \mathbb{C})$ the Poincare dual of $H^{2,0}(X)$. Then, we have

$$\alpha_{\mathbb{C}}(H) \in \Omega.$$

Definition 12 (Periods). We say $\alpha_{\mathbb{C}}(H)$ the **period** of a marked real 2-elementary K3 surface $(X, \tau, \varphi, \alpha)$ of type (S, θ) satisfying $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$. \square

Remark 8. A point in Ω is not necessarily the period of a marked real 2-elementary K3 surface of type (S, θ) satisfying $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$. Moreover, of course, it is not necessarily the period of some marked real 2-elementary K3 surface of type (S, θ) satisfying $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$ **which has some required properties.** \square

Definition 13 (Equivalence). We say two points $[\omega], [\omega'] (\in \Omega)$ are **equivalent** if one of two is obtained by the other from an automorphism of (\mathbb{L}_{K3}, ψ) with respect to the group G . (Recall Definition 10.) \square

Remark 9. Suppose that $[\omega]$ is equivalent to $[\omega']$. If $[\omega]$ is the period of some marked real 2-elementary K3 surface of type (S, θ) satisfying $\alpha \circ \varphi_* \circ \alpha^{-1} = \psi$, then so $[\omega']$ is. \square

Let us describe the domain Ω . It has two connected components which are interchanged by $-\psi$. Since $-\psi$ is an automorphism of (\mathbb{L}_{K3}, ψ) with respect to the group G , it is sufficient that we investigate the quotient space

$$\Omega / -\psi$$

from the point of view of the “equivalence” (Remark 9).

For (\mathbb{L}_{K3}, ψ) , we set

$$\mathbb{L}_{\pm} := \{x \in \mathbb{L}_{K3} \mid \psi(x) = \pm x\}.$$

Obviously, \mathbb{L}_{\pm} depend on the integral involution ψ .

For $[\omega] \in \Omega$ ($\omega \in \mathbb{L}_{K3} \otimes \mathbb{C}$), we have the decomposition

$$\omega = \omega_+ + \omega_-,$$

where $\omega_{\pm} \in \mathbb{L}_{\pm} \otimes \mathbb{R}$.

We now restrict ourselves the case $S \subset \mathbb{L}_-$, namely, $\theta = -1$. and suppose that $G = \{1_S\}$.

We set

$$\mathbb{L}_{-,S} := \mathbb{L}_- \cap S^{\perp}.$$

Since $\omega_- \in \mathbb{L}_{-,S} \otimes \mathbb{R}$ and $\omega_+^2 = \omega_-^2 > 0$, we see that $\mathbb{L}_+, \mathbb{L}_{-,S}$ is a **hyperbolic** lattice, i.e., has one positive square.

Let $\mathcal{L}_+, \mathcal{L}_{-,S}$ be the Lobachevsky spaces (hyperbolic spaces) obtained from $\mathbb{L}_+ \otimes \mathbb{R}, \mathbb{L}_{-,S} \otimes \mathbb{R}$ respectively. Then we have

$$\Omega / -\psi = \mathcal{L}_+ \times \mathcal{L}_{-,S} \quad (\text{a direct product}).$$

We shall use this domain $\Omega / -\psi$ for $(S, \theta) = (\langle 2 \rangle \oplus \langle -2 \rangle, -1), ((3, 1, 1), -1)$ later.

3. 2-ELEMENTARY K3 SURFACES OF TYPE $(3, 1, 1)$

In this paper we especially study real 2-elementary K3 surfaces of type $((r(S), a(S), \delta(S)), \theta) = ((3, 1, 1), -1)$.

Fact 1 (Alexeev and Nikulin [1]). *Let (X, τ) be a 2-elementary K3 surface of type $S = (3, 1, 1)$. We have the following.*

- The fixed point curve $A := X^\tau$ is contained in $|-2K_X|$, and

$$A = A_0 \cup A_1 \quad (\text{disjoint union}),$$

where A_0 is a nonsingular rational curve $(\cong \mathbb{P}^1)$ with $A_0^2 = -2$ and A_1 is a nonsingular curve of genus 9.

- (X, τ) has a structure of an **elliptic pencil** $|E + F|$ with **its section** A_0 and a **unique reducible fiber** $E + F$ satisfying the following conditions:
 - (i): E is a nonsingular rational curve with $E^2 = -2$ and $E \cdot A_0 = 1$.
 - (ii): $E \cdot F = 2$, $F^2 = -2$, $F \cdot A_0 = 0$, and F is either
 - (1) a nonsingular rational curve ($\widetilde{\mathbb{A}}_1$ type), or
 - (2) the union of two nonsingular rational curves F' and F'' which are conjugate by τ and $F' \cdot F'' = 1$. ($\widetilde{\mathbb{A}}_2$ type)
 - (iii): The elliptic pencil does not have other reducible fibres.
 - (iv): τ is the inverse map (of the group) on each elliptic fibre.
 - (v): A_0 , E and F generate the lattice $H_{2+}(X, \mathbb{Z}) (\cong S)$. \square

Moreover, we have $A_1 \cdot E = 1$, $A_1 \cdot F = 2$. Thus we have the Gram matrix of the lattice S with respect to the basis E , F and A_0 of $H_{2+}(X, \mathbb{Z}) (\cong S)$:

$$\begin{array}{ccc} & E & F & A_0 \\ E & -2 & & \\ F & 2 & -2 & \\ A_0 & 1 & 0 & -2 \end{array}$$

Hence, the discriminant group S^*/S is a 2-elementary group of rank $a(S) = 1$. The element $F/2 \in S^*$ has $(F/2)^2 = -1/2 \notin \mathbb{Z}$. Hence, we have $\delta(S) = 1$.

We consider the quotient surface

$$Y := X/\{1, \tau\}$$

(so-called ‘‘DPN surface’’, see [1] or [9]) and let

$$\pi : X \rightarrow Y$$

be the quotient map. We set

$$e := \pi(E) \quad \text{and} \quad f := \pi(F).$$

Here, if F is of $\widetilde{\mathbb{A}}_2$ type (see Fact 1), i.e., F is the union of two nonsingular rational curves F' and F'' which are conjugate by τ and $F' \cdot F'' = 1$, then $f = \pi(F) = \pi(F' \cup F'') = \pi(F') = \pi(F'')$.

Since

$$A = A_0 \cup A_1 \quad (\text{disjoint union})$$

is the fixed point curve of τ , we use the same symbols A_0 and A_1 for their images in Y by π .

Then, the Picard group $\text{Pic}(Y)$ of Y is generated by the curves e , f and A_0 . We have the Gram matrix of $\text{Pic}(Y)$ with respect to the basis e , f and A_0 as follows:

$$\begin{array}{ccc} & e & f & A_0 \\ e & -1 & & \\ f & 1 & -1 & \\ A_0 & 1 & 0 & -4 \end{array}$$

Remark 10 ([10]). *For any 2-elementary K3 surface (X, τ) of type $(3, 1, 1)$, all exceptional curves on the quotient surface Y are exactly the curves e , f and A_0 . Hence, Y has no exceptional curves with square (-2) , and (X, τ) is “**D-nondegenerate**” in the sense of [9]. \square*

Moreover, for the curve A_1 , we have

$$A_1 \cdot e = 1 \quad \text{and} \quad A_1 \cdot f = 2.$$

Remark 11 (cf. Fact 1 (ii), see also Remark 12 below.). *We have the following.*

- (1) *F is a nonsingular rational curve on X . $\iff A_1$ intersects f in two distinct points on Y .*
- (2) *F is a union of two nonsingular rational curves on X . $\iff A_1$ touches f on Y . \square*

4. REAL 2-ELEMENTARY K3 SURFACES OF TYPE $((3, 1, 1), -1)$

We now consider **real** 2-elementary K3 surfaces (X, τ, φ) of type $((3, 1, 1), \theta)$. We use the same notation as the preceding section. As in [10], the induced anti-holomorphic involution $\varphi_{\text{mod } \tau}$ on Y should send each of A_0 , e and f to itself changing its orientation. Thus we have

$$\theta = -1.$$

Contracting the exceptional curve $f = \pi(F)$ on Y to a point, we get a map onto the 4-th Hirzebruch surface

$$\text{bl} : Y \rightarrow \mathbb{F}_4.$$

We have

$$\text{bl}(A) = \text{bl}(A_0) + \text{bl}(A_1) \in |-2K_{\mathbb{F}_4}|$$

and we set

$$s := \text{bl}(A_0), \quad A'_1 := \text{bl}(A_1), \quad c := \text{bl}(e).$$

Namely,

$$\text{bl}(A) = s + A'_1 \in |-2K_{\mathbb{F}_4}|.$$

Then s is the exceptional section of \mathbb{F}_4 with $s^2 = -4$, and c is a fiber of the fibration $\mathbb{F}_4 \rightarrow s$. Since $s \cdot A'_1 = 0$, A'_1 does not intersect the section s . One has $c^2 = 0$. We have

$$-2K_{\mathbb{F}_4} \sim 12c + 4s.$$

It follows

$$A'_1 \in |12c + 3s|.$$

Remark 12 (cf. Remark 11). *Since $A_1 \cdot f = 2$ in Y , the curve A'_1 has **one real double point**.*

- (1) *A_1 intersects with f at two distinct points¹ in Y .*

\iff *The double point of A'_1 is **nondegenerate**.*

- (2) *A_1 touches to f in Y .*

\iff *The double point of A'_1 is **degenerate**.*

See Figure 1 in Section 6 below. \square

Remark 13 (Blow down of Y to \mathbb{F}_3 , see [10]). *If we contract the exceptional curve $e = \pi(E)$ of Y to a point, then we get a map onto the 3-th Hirzebruch surface \mathbb{F}_3 :*

$$\text{bl}_1 : Y \rightarrow \mathbb{F}_3$$

Then $s := \text{bl}_1(A_0)$ is the exceptional section of \mathbb{F}_3 with $s^2 = -3$ and $c := \text{bl}_1(f)$ is a fiber. Since $\text{bl}_1(A) = s + \text{bl}_1(A_1) \in |-2K_{\mathbb{F}_3}|$ and $K_{\mathbb{F}_3} \sim -2s - 5c$, we have $\text{bl}_1(A) \sim 4s + 10c$. And

$$A_1 := \text{bl}_1(A_1) \quad (\sim 3s + 10c)$$

*is a **nonsingular** curve of genus 9. The classification of A_1 was done in [10]. \square*

¹ They are two different real points (\rightarrow a real node) or two different non-real conjugate points (\rightarrow a real isolated point).

5. ENUMERATION OF REAL 2-ELEMENTARY K3 SURFACES OF TYPE $((3, 1, 1), -1)$

As mentioned in Remark 10, for any 2-elementary K3 surface (X, τ) of type $(3, 1, 1)$, the quotient surface Y has no exceptional curves with square -2 and hence, (X, τ) is **\mathcal{D} -nondegenerate** in the sense of [9]. For any **real** 2-elementary K3 surface (X, τ, φ) of type $(S, \theta) = ((3, 1, 1), -1)$ (recall Section 4), we have $S = S_-$ and hence, (X, τ, φ) is **\mathcal{DR} -nondegenerate** in the sense of [9].

Remark 14. $G = \{1_S\}$ for the case $(S, \theta) = ((3, 1, 1), -1)$ (recall Definition 7). \square

Therefore, by Theorem 1 of [9], the connected components of the moduli of real 2-elementary K3 surfaces (X, τ, φ) of type $((3, 1, 1), -1)$ are in one to one (bijective) correspondence with the **isometry classes** of involutions of \mathbb{L}_{K3} of type $((3, 1, 1), -1)$.

Moreover, by Theorem 14 and Proposition 15 of [9], each genus determines the isometry class. The complete genus invariants can be enumerated by the more general results of Nikulin [8].

Remark that we have an orthogonal decomposition

$$S \cong \mathbb{U} \oplus \mathbb{Z}(F)$$

with respect to the basis $A_0 + E + F$, $E + F$, and F , where \mathbb{U} is the hyperbolic even unimodular lattice of signature $(1, 1)$, and $\mathbb{Z}(F) \cong \langle -2 \rangle$.

Hence, the enumeration can be reduced to the invariants of “polarized integral involutions” ([6], §3). In this case all necessary calculations are already done in Theorem 3.4.3 of [6]. Since $\tau_*|_{\mathbb{U}} = 1$ and $\varphi_*|_{\mathbb{U}} = -1$, a triplet $(H_2(X, \mathbb{Z}), \tau_*, \varphi_*)$ defines a polarized integral involution (L_1, φ_1, F) , where $L_1 := \mathbb{U}^\perp$ in $H_2(X, \mathbb{Z})$, and it is an even unimodular lattice of signature $(2, 18)$, $\varphi_1 = \varphi_*|_{L_1}$ and $L_1^{\varphi_1}$ is hyperbolic, $\varphi_1(F) = -F$ and $F^2 = -2$.

We temporary set

$$(L, \tau, \varphi) := (H_2(X, \mathbb{Z}), \tau_*, \varphi_*).$$

The complete **isometry invariants** of the integral involution (L, τ, φ) are

$$(5.1) \quad (r(\varphi), a(\varphi), \delta_\varphi, \delta_{\varphi F}, \delta_F(\varphi)),$$

where $r(\varphi) := \text{rank } L^\varphi \in \mathbb{N}$; $((L^\varphi)^*/L^\varphi) = (\mathbb{Z}/2\mathbb{Z})^{a(\varphi)}$ where $a(\varphi) \geq 0$ is an integer; $\delta_\varphi \in \{0, 1\}$ is equal to 0 if and only if $x \cdot \varphi(x) \equiv 0 \pmod{2}$ for any $x \in L$; $\delta_{\varphi F} \in \{0, 1\}$ is equal to 0 if and only if $x \cdot \varphi(x) \equiv x \cdot F \pmod{2}$ for any $x \in L$; and $\delta_F(\varphi) \in \{0, 1\}$ is equal to 0 if and only if $x \cdot F \equiv 0 \pmod{2}$ for any $x \in L_\varphi$.

Instead of δ_φ , $\delta_{\varphi F}$ and $\delta_F(\varphi)$, we also have other invariants: the subgroup

$$H(\varphi) \quad (\subset \mathbb{Z}/2\mathbb{Z}(F) = 2S^*/2S),$$

the invariant

$$\delta_{\varphi S} \in \{0, 1\},$$

and the “characteristic element” ([8])

$$v \in H(\varphi)$$

when $\delta_{\varphi S} = 0$. The v is not defined when $\delta_{\varphi S} = 1$. They are related as follows: $H(\varphi) = 0$ if and only if $\delta_F(\varphi) = 1$; $\delta_\varphi = 0$ if and only if $\delta_{\varphi S} = 0$ and $v = 0 \pmod{2S}$; $\delta_{\varphi F} = 0$ if and only if $\delta_{\varphi S} = 0$ and $v \sim F \pmod{2S}$; and $\delta_{\varphi S} = 1$ if and only if $\delta_\varphi = \delta_{\varphi F} = 1$.

Note that by Conditions 1.8.1 of [8], $\delta_\varphi = 0$ implies $H(\varphi) = 0$. Hence, if $H(\varphi) = \mathbb{Z}/2\mathbb{Z}(F)$ and $\delta_{\varphi S} = 0$, then $\delta_\varphi = 1$, equivalently, $v = [F]$.

Thus we get the following correspondence:

δ_φ	$\delta_{\varphi F}$	$\delta_F(\varphi)$	$\delta_{\varphi S}$	$H(\varphi)$
0	1	1	0	0
1	1	1	1	0
1	0	0	0	$\mathbb{Z}/2\mathbb{Z}(F)$
1	1	0	1	$\mathbb{Z}/2\mathbb{Z}(F)$

Thus, the data (5.1) is just equivalent to the data

$$(5.2) \quad (r(\varphi), a(\varphi), \delta_{\varphi S}, H(\varphi)).$$

For all possible data (5.2), see **Table 1** ($H(\varphi) = 0$ case) and **Table 2** ($H(\varphi) = \mathbb{Z}/2\mathbb{Z}(F)$ case) **where 0 or 1 in each cell stands for the value of $\delta_{\varphi S}$.**

There are 12 data of Type 0 ($\Rightarrow H(\varphi) = 0$), 12 data of Type Ia, 39 data of Type Ib with $H(\varphi) = 0$, and 39 data of Type Ib with $H(\varphi) = \mathbb{Z}/2\mathbb{Z}(F)$.

9									1									
8								1		0, 1								
7							1		1		1							
6						1		1		0, 1		1						
5					1		1		1		1		1					
4				1		0, 1		1		0, 1		1		0, 1				
3			1		1		1		1		1		1		1			
2		0, 1		1		0		1		0, 1		1		0		1		
1	1		1						1		1						1	
0		0								0								0
a / r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

TABLE 1. All possible data $(r, a, \delta_{\varphi S})$ for $H(\varphi) = 0$.

10										1								
9									0, 1		1							
8							1		1		1							
7							1		0, 1		1		1					
6						1		1		1		1		1				
5					0, 1		1		0, 1		1		0, 1		1			
4				1		1		1		1		1		1		1		
3			1		0		1		0, 1		1		0		1		0, 1	
2		1						1		1						1		1
1	0								0								0	
a / r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

TABLE 2. All possible data $(r, a, \delta_{\varphi S})$ for $H(\varphi) = \mathbb{Z}/2\mathbb{Z}(F)$.

By simple direct calculations or applying [9], Theorem 11 and Corollary 13), one finds how invariants (5.1) of φ and those of its “related involution ([9])” $\tilde{\varphi} = \tau \circ \varphi$ are calculated from each other:

$$\begin{aligned} r(\varphi) + r(\tilde{\varphi}) &= 19; \quad a(\varphi) + \delta_F(\varphi) = a(\tilde{\varphi}) + \delta_F(\tilde{\varphi}); \\ \delta_F(\varphi) + \delta_F(\tilde{\varphi}) &= 1; \quad \delta_{\tilde{\varphi}} = \delta_{\varphi F}; \quad \delta_{\tilde{\varphi} F} = \delta_{\varphi}. \end{aligned}$$

Theorem 1 ([9], [10]). *There are exactly **102** connected components of the moduli of real 2-elementary K3 surfaces of type $((3, 1, 1), -1)$. Identifying “related” pairs ([9]) of anti-holomorphic involutions on the K3 surfaces, there are exactly **51** connected components.*² \square

We mention about some topological interpretations of genus invariants.

Definition 14 (Dividing curves). *In general, let A be a possibly disconnected nonsingular complex compact curve with an anti-holomorphic involution φ , and $\mathbb{R}A$ the fixed point set of φ . We say (A, φ) (or simply, A) is **dividing** (or **type I**) if $[\mathbb{R}A] = 0$ in $H_1(A; \mathbb{Z}/2\mathbb{Z})$.* \square

²The moduli of real nonsingular curves $A \in |-2K_{\mathbb{F}_3}|$ up to the action of $Aut(\mathbb{F}_3/\mathbb{R})$.

Theorem 2 ([10]). *Let (X, τ, φ) be a real 2-elementary K3 surface of type (S, θ) , and $A = X^\tau$ be the fixed point set of τ in X . Assume that A is non-empty. Then, the real curve A is dividing if and only if $\delta_{\varphi S} = 0$. \square*

For $((3, 1, 1), -1)$ case, we have $A = A_0 \cup A_1$ where $A_0 \cong \mathbb{P}^1$ and A_1 has genus 9. Since A_0 is always dividing, A is dividing if and only if A_1 is dividing. Then Theorem 1 is rephrased as follows:

Theorem 3 ([10]). *The connected component of the moduli of real 2-elementary K3 surfaces of type $((3, 1, 1), -1)$ is determined by the **isotopy type** of the half domain³ $A_- := \pi(X_\varphi(\mathbb{R})) \subset Y(\mathbb{R})$ and the dividingness of the curve A_1 . \square*

6. REAL PARTS OF REAL 2-ELEMENTARY K3 SURFACES X OF TYPE $((3, 1, 1), -1)$ AND THEIR QUOTIENT SURFACES Y

6.1. The half domains A_+ and A_- in $Y(\mathbb{R})$. Let (X, τ, φ) be a real 2-elementary K3 surface of type $((3, 1, 1), -1)$. We use the notation in Section 4. Let $X_\varphi(\mathbb{R})$ denote the real part (the fixed point set of φ) of the real K3 surface (X, φ) . Note that $X_\varphi(\mathbb{R})$ is **always non-empty** ([10]).

Let $Y(\mathbb{R})$ be the real part of the quotient surface (DPN surface) Y with the real structure $\varphi_{\text{mod } \tau}$. The real part $\mathbb{R}A = \mathbb{R}A_0 \cup \mathbb{R}A_1$ of the branch curve A divides $Y(\mathbb{R})$ into two half domains

$$A_+, \quad A_-.$$

Of course, each half domain A_\pm might have several connected components. One of the half domains A_+ and A_- is doubly covered by the real part $X_\varphi(\mathbb{R})$, and the other by the real part $X_{\tilde{\varphi}}(\mathbb{R})$, where we set $\tilde{\varphi} := \tau \circ \varphi$. We call ([9]) $\tilde{\varphi}$ the related involution of φ .

Remark 15 (Topological interpretation of $\delta_F(\varphi)$, [10]). *Recall that there are two kinds of curves F , which are irreducible and reducible (Fact1, Remark11, and Remark12). But we can always take the curve F irreducible ($\cong \mathbb{P}^1$) in the same connected component of the moduli ([9]) of real 2-elementary K3 surfaces of type $((3, 1, 1), -1)$, equivalently, in the same isometry class of involutions of \mathbb{L}_{K3} of type $((3, 1, 1), -1)$. \square*

Since $X_\varphi(\mathbb{R})$ is non-empty, by [9], we have:

Lemma 4. *If we take the curve F irreducible, then we have*

$$\delta_F(\varphi) = 0 \iff [F_\varphi(\mathbb{R})] = 0 \text{ in } H_1(X_\varphi(\mathbb{R}); \mathbb{Z}/2\mathbb{Z}).$$

We **distinguish** the two half domains A_+ and A_- using F as follows.

Definition 15 (The half domains A_+ and A_-). *For a real 2-elementary K3 surface (X, τ, φ) of type $((3, 1, 1), -1)$, we define the half domains A_+ and $A_- \subset Y(\mathbb{R})$ as follows:*

$$\begin{cases} A_+ := \pi(X_\varphi(\mathbb{R})) & \text{if } \delta_F(\varphi) = 0 \quad (\Leftrightarrow [F_\varphi(\mathbb{R})] = 0 \text{ in } H_1(X_\varphi(\mathbb{R}); \mathbb{Z}/2\mathbb{Z})), \\ A_- := \pi(X_\varphi(\mathbb{R})) & \text{if } \delta_F(\varphi) = 1 \quad (\Leftrightarrow [F_\varphi(\mathbb{R})] \neq 0 \text{ in } H_1(X_\varphi(\mathbb{R}); \mathbb{Z}/2\mathbb{Z})). \end{cases}$$

Note that if $\delta_F(\varphi) = 0$, then $\delta_F(\tilde{\varphi}) = 1$.

6.2. Real curves $\mathbb{R}A'_1$ in $|12c + 3s|$ with one real double point on $\mathbb{R}\mathbb{F}_4$. Recall (Section 4) that $s := \text{bl}(A_0)$ is the exceptional section with $s^2 = -4$, and $c := \text{bl}(e)$ is a fiber of \mathbb{F}_4 ,

$$A'_1 := \text{bl}(A_1) \subset \mathbb{F}_4 \quad \text{and} \quad A'_1 \in |12c + 3s|$$

The curve f is contracted to a point on the fiber c via the map bl . We set

$$P_0 := \text{bl}(f). \quad (\text{the contracting point})$$

As stated above, P_0 is a unique (possibly degenerate) double point of A'_1 .

There are three real isotopy types of the double point P_0 (cf. Remark12):

³It is called the “**positive curve**” in [9]. See also Subsection 6.1 below.

Node case: If A_1 intersects with f at two different real points, then the real point P_0 is a **node** of $\mathbb{R}A'_1$.

Cusp case: If A_1 touches to f at one real point, then the real point P_0 is a **cusp** of $\mathbb{R}A'_1$. (degenerate double point)

Isolated point case: If A_1 intersects with f at two different non-real conjugate points, then the two points commute by $\varphi_{\text{mod } \tau}$ and the real point P_0 is an **isolated point** of $\mathbb{R}A'_1$.

See Figure 1 below.

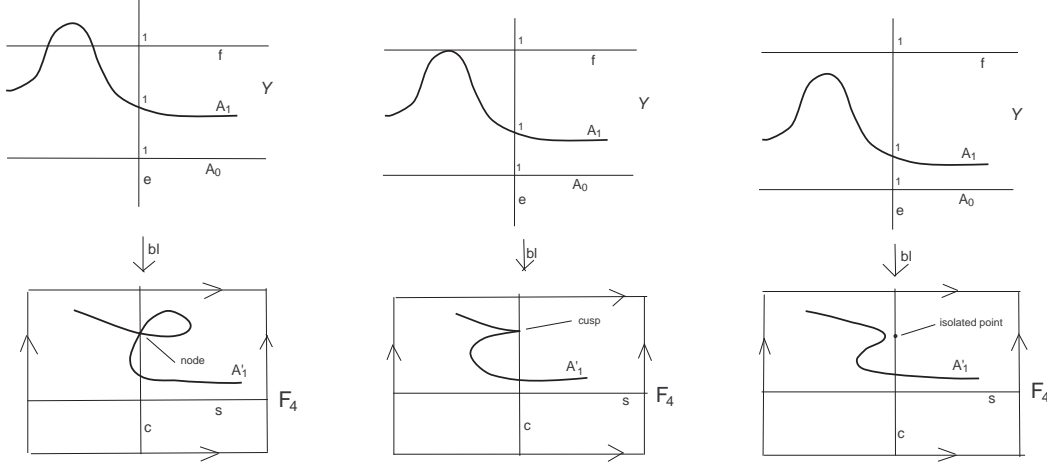


FIGURE 1. The real double point P_0 of the curve A'_1

Due to [9], we possess the moduli (up to $\text{Aut}(\mathbb{F}_4/\mathbb{R})$) of real curves in $|-2K_{\mathbb{F}_4}|$ with one real double point. However, we see that **the connected components of this moduli ([9]) can not distinguish the real isotopy types (nodes, cusps, or isolated points)** of these real double points. We want to classify the real isotopy types of these real double points P_0 of the curves $\mathbb{R}A'_1$.

We have $A'_1 \cdot s = 0$ and $A'_1 \cdot c = 3$. Since $f \cdot A_0 = 0$ in Y , the section s does not meet the point P_0 . We may assume that e does not pass through any intersection point of A_1 and f in Y . (See Figure 1.) Since $A_1 \cdot e = 1$ in Y , the intersection point of A_1 with e is **real** and does not meet f . Via the map bl , this point goes to a real intersection point of A'_1 with c with multiplicity 1. Thus we set

$$P_1 := \text{bl}(A_1 \cap e). \quad (\text{the intersection point})$$

Since $A'_1 \cdot c = 3$, A'_1 intersects with c at P_0 with multiplicity 2 and at P_1 with multiplicity 1.

Since $A'_1 \cap s = \emptyset$, any **non-contractible (possibly real singular)** components of the real part $\mathbb{R}A'_1$ are “parallel” to $\mathbb{R}s$. See Figure2, Figure3 and Figure4.

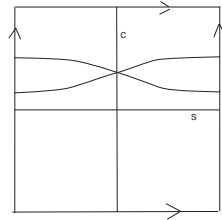


FIGURE 2. Non-contractible component with **Node (*)**

We call a connected component of $\mathbb{R}A'_1$ an **oval** if it has no real singular points and contractible in $\mathbb{R}\mathbb{F}_4 \simeq T^2$ (2-torus), that is, realizes 0 in $H_1(\mathbb{R}\mathbb{F}_4; \mathbb{Z})$. Since $A'_1 \cdot c = 3$, the interior of each oval of $\mathbb{R}A'_1$ does not contain any other ovals.

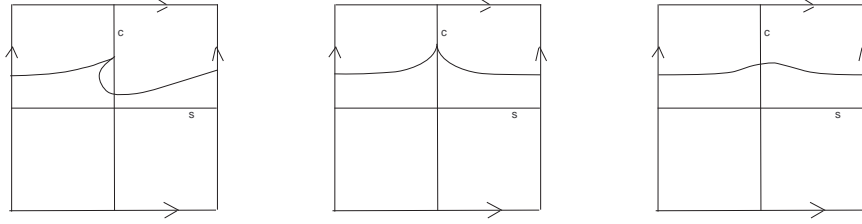


FIGURE 3. Non-contractible component with a cusp or without singular points

We now classify **all possible cases**.

NODE case:

Node (1) case: We consider the case when **both** the node P_0 and the intersection point P_1 are contained in **the same** connected (singular) component of $\mathbb{R}A'_1$. Then $\mathbb{R}A'_1$ meets $\mathbb{R}c$ at only P_0 and P_1 , and the component above is **not contractible**. See Figure 4. $\mathbb{R}A'_1$ may have some ovals.

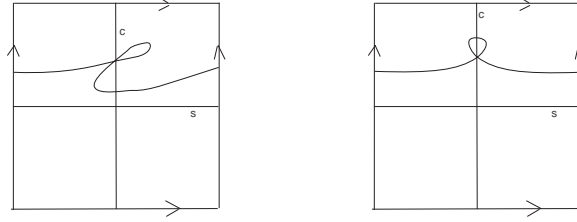


FIGURE 4. Non-contractible component with **Node (1)**

Since $A'_1 \cdot c = 3$, the interior of any oval of $\mathbb{R}A_1$ does not contain other ovals. The interior of the node also does not contain ovals.

The non-contractible component containing P_0 and P_1 and the section $\mathbb{R}s$ divide $\mathbb{R}F_4$ (2-torus) into three parts, which are **the interior of the node and **two noncontractible domains**.**

Definition 16 (The regions R_1 and R_2 in Node (1) case). *Let R_1 denote **the noncontractible region which is connected with the interior of the node in the blow up of $\mathbb{R}F_4$** , and let R_2 denote the other noncontractible region. We define the numbers α and β as follows:*

$$(6.1) \quad \begin{aligned} \alpha &:= \#\{\text{ovals contained in } R_1\}, \\ \beta &:= \#\{\text{ovals contained in } R_2\}. \end{aligned}$$

See the left figure of Figure 5. \square

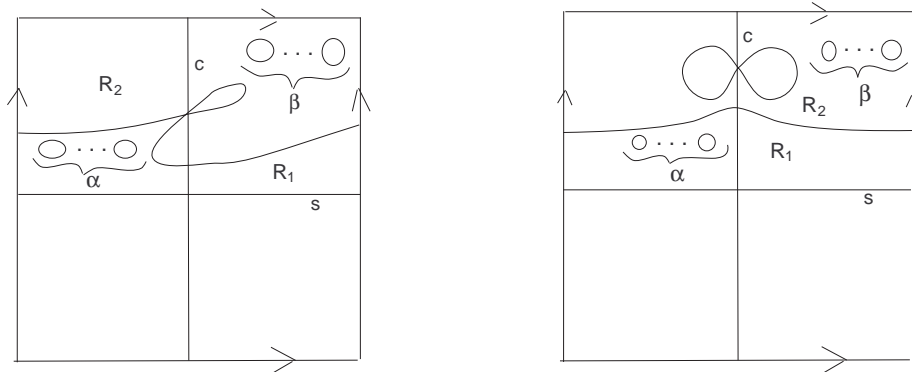


FIGURE 5. **Node (1)** and **Node (2)**, The regions R_1 and R_2

Node (2) case and Node (*) case: When the node P_0 and the intersection point P_1 respectively are contained in **different** connected components of $\mathbb{R}A'_1$, the component containing P_1 is nonsingular and not contractible like the rightmost figure of Figure 3 above. The component containing P_0 can be either **contractible** (the left figure of Figure 6) or **non-contractible** (Figure 2).

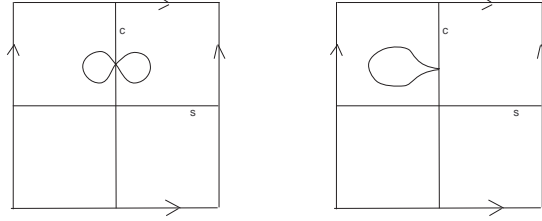


FIGURE 6. Contractible component with a node or a cusp

Node (2) case: When the component containing the node P_0 is **contractible** (the left figure of Figure 6), $\mathbb{R}A'_1$ might have some ovals. **The component containing P_1 and the section $\mathbb{R}s$** divide $\mathbb{R}F_4$ into two regions.

Definition 17 (The regions R_1 and R_2 in Node (2) case). *Let R_1 denote the region which does not contain the contractible component containing the node P_0 , and let R_2 denote the other region. (Since $A'_1 \cdot c = 3$, the interior of the contractible component containing P_0 and the interior of any oval of $\mathbb{R}A_1$ cannot contain any other ovals.) We define the numbers α and β by (6.1). See the right figure of Figure 5. \square*

Node (*) case: When the component containing the node P_0 is **non-contractible** (Figure 2), $\mathbb{R}A'_1$ has **no ovals** (Figure 7).

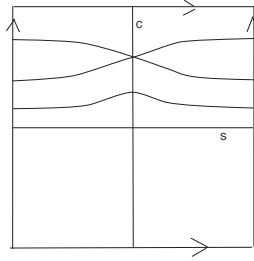


FIGURE 7. Node (*)

CUSP case:

Cusp (1): When both the cusp P_0 and the point P_1 are contained in **the same** connected component of $\mathbb{R}A'_1$, $\mathbb{R}A'_1$ meets $\mathbb{R}c$ at only P_0 and P_1 , and the component containing P_0 and P_1 is not contractible (the leftmost figure of Figure 3). $\mathbb{R}A'_1$ might have some ovals. **The non-contractible component containing P_0 and P_1 and the section $\mathbb{R}s$** divide $\mathbb{R}F_4$ into two regions (the left figure of Figure 8).

One of these regions goes to a non-orientable region via the blow up of $\mathbb{R}F_4$.

Definition 18 (The regions R_1 and R_2 in Cusp (1) case). *Let R_2 denote the region which goes to a non-orientable region via the blow up of $\mathbb{R}F_4$. and let R_1 denote the other region. (Since $A'_1 \cdot c = 3$, the interior of any oval of $\mathbb{R}A_1$ does not contain any other ovals.) We define the numbers α and β by (6.1). See the left figure of Figure 8. \square*

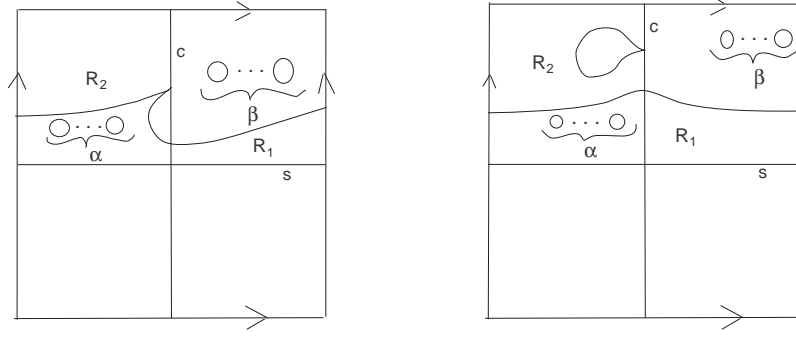


FIGURE 8. **Cusp (1)** and **Cusp (2)**

Cusp (2) When the cusp P_0 and P_1 respectively are contained in **different** connected components of $\mathbb{R}A'_1$, the component containing P_1 is not contractible (the right figure of Figure 3). The component containing the cusp P_0 should be **contractible** (the right figure of Figure 6). (If not, then this component would be like the middle figure of Figure 3 and the number of the intersection points of $\mathbb{R}A'_1$ with $\mathbb{R}c$ would be even. This contradicts with $A'_1 \cdot c = 3$.) $\mathbb{R}A'_1$ might have some ovals. **The component containing P_1 and the section $\mathbb{R}s$ divide $\mathbb{R}F_4$ into two regions.**

Definition 19 (The regions R_1 and R_2 in Cusp (2) case). *Let R_1 denote the region which does not contain the contractible component which contains the cusp P_0 , and let R_2 denote the other region. (the right figure of Figure 8) (Since $A'_1 \cdot c = 3$, the interior of any oval of $\mathbb{R}A_1$ does not contain any ovals.) We define the numbers α and β by (6.1). See the right figure of Figure 8 above. \square*

ISOLATED POINT case:

In this case the connected component containing P_1 is nonsingular and non-contractible like the right figure of Figure 3. $\mathbb{R}A'_1$ might have some ovals. **The component containing P_1 and the section $\mathbb{R}s$ divide $\mathbb{R}F_4$ into two regions.**

Definition 20 (The regions R_1 and R_2 in Isolated point case). *Let R_1 denote the region which does not contain the isolated point, and let R_2 denote the other region. (Since $A'_1 \cdot c = 3$, the interior of any oval of $\mathbb{R}A_1$ does not contain any other ovals.) We define the numbers α and β by (6.1). See Figure 9. \square*

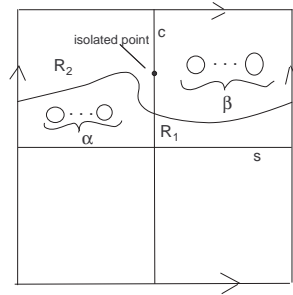


FIGURE 9. **Isolated point**

Theorem 5. *From the above arguments, there are 6 types of curves $\mathbb{R}A'_1$ in $|12c + 3s|$ with one double point on $\mathbb{R}F_4$ as in Table 3. \square*

I	<i>Node (1)</i>	<i>Cusp (1)</i>	<i>Isolated point</i>
II	<i>Node (2)</i>	<i>Cusp (2)</i>	
III	<i>Node (*)</i>		

TABLE 3. All cases for real curves with one real double point in $|-2K_{\mathbb{F}_4}|$ on $\mathbb{R}F_4$

6.3. The real parts of K3 surfaces X and their quotient surfaces Y via the blow up of \mathbb{F}_4 . Now we determine the topology of the real parts $X_\varphi(\mathbb{R})$ and $X_{\tilde{\varphi}}(\mathbb{R})$ of a real 2-elementary K3 surface (X, τ, φ) of type $((3, 1, 1), -1)$, and the real part $Y(\mathbb{R})$ of the quotient surface Y with the real structure $\varphi_{\text{mod } \tau}$.

Recall each cases in Table 3.

I. Node (1), Cusp (1) and Isolated point cases.

In these cases, by the definitions of the half domains A_\pm and R_1, R_2 , we see that

- A_+ is homeomorphic to the disjoint union of (an annulus with α holes) and (β disks), and
- A_- is homeomorphic to the disjoint union of (((an annulus $\setminus D^2$) \cup **Möbius band**) with β holes) and (α disks).

For example, see Figure 10 for **Node case (1)**.

Suppose that the invariant $\delta_F(\varphi) = 1$ for the involution φ . We take F to be **irreducible** as mentioned above. Then we have $[X_\varphi(\mathbb{R}) \cap F] \neq 0$ and $\pi(X_\varphi(\mathbb{R})) = A_-$. On the other hand, for $\tilde{\varphi}$, we have $[X_{\tilde{\varphi}}(\mathbb{R}) \cap F] = 0$ and $\pi(X_{\tilde{\varphi}}(\mathbb{R})) = A_+$.

We can say that

$$\begin{aligned} \alpha &= \#\{\text{ovals whose interiors are contained in } \text{bl}(A_-)\}, \text{ and} \\ \beta &= \#\{\text{ovals whose interiors are contained in } \text{bl}(A_+)\}. \end{aligned}$$

Thus we have

$$X_\varphi(\mathbb{R}) \sim \Sigma_{2+\beta} \cup \alpha S^2.$$

Moreover, we have $(r(\varphi), a(\varphi), \delta(\varphi)) \neq (10, 10, 0), (10, 8, 0),$

$$r(\varphi) = 9 + \alpha - \beta, \quad a(\varphi) = 9 - \alpha - \beta, \quad \text{and} \quad H(\varphi) = 0.$$

because $\delta_F(\varphi) = 1$. On the other hand, we have

$$X_{\tilde{\varphi}}(\mathbb{R}) \sim \Sigma_{1+\alpha} \cup \beta S^2.$$

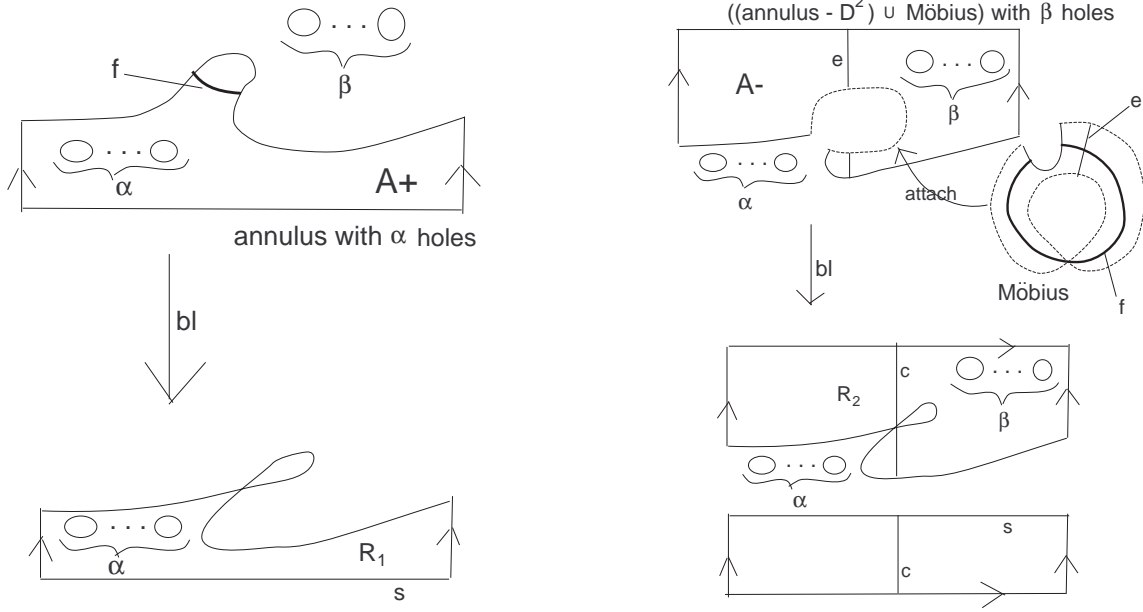


FIGURE 10. The half domains A_+ and A_- . **Node case (1)**

Hence, we have $(r(\tilde{\varphi}), a(\tilde{\varphi}), \delta(\tilde{\varphi})) \neq (10, 10, 0), (10, 8, 0)$,

$$r(\tilde{\varphi}) = 10 - \alpha + \beta, \quad a(\tilde{\varphi}) = 10 - \alpha - \beta, \quad \text{and} \quad H(\tilde{\varphi}) = \mathbb{Z}/2\mathbb{Z}(F).$$

(We omit the cusp case and isolated point case.)

II. Node (2) and Cusp (2) cases.

In these cases, by the definitions of the half domains A_{\pm} and R_1, R_2 , we see that

- A_+ is homeomorphic to the disjoint union of (an annulus with α holes) and $((\beta + 1)$ disks), and
- A_- is homeomorphic to the disjoint union of $((\text{an annulus} \setminus D^2) \cup \text{Möbius band})$ with $(\beta + 1)$ holes) and $(\alpha$ disks).

For example, See Figure 11 for **Node (2)**.

Suppose that $A_- = \pi(X_{\varphi}(\mathbb{R}))$, namely, the invariant $\delta_F(\varphi) = 1$ for the involution φ . Then we have

$$X_{\varphi}(\mathbb{R}) \sim \Sigma_{2+(\beta+1)} \cup \alpha S^2.$$

Moreover, we have $(r(\varphi), a(\varphi), \delta(\varphi)) \neq (10, 10, 0), (10, 8, 0)$,

$$r(\varphi) = 8 + \alpha - \beta, \quad a(\varphi) = 8 - \alpha - \beta, \quad \text{and} \quad H(\varphi) = 0$$

because $\delta_F(\varphi) = 1$.

On the other hand, we have $A_+ = \pi(X_{\tilde{\varphi}}(\mathbb{R}))$ and

$$X_{\tilde{\varphi}}(\mathbb{R}) \sim \Sigma_{1+\alpha} \cup (\beta + 1) S^2,$$

Moreover, we have $(r(\tilde{\varphi}), a(\tilde{\varphi}), \delta(\tilde{\varphi})) \neq (10, 10, 0), (10, 8, 0)$,

$$r(\tilde{\varphi}) = 11 - \alpha + \beta, \quad a(\tilde{\varphi}) = 9 - \alpha - \beta, \quad \text{and} \quad H(\tilde{\varphi}) = \mathbb{Z}/2\mathbb{Z}(F).$$

(Here we omit the cusp cases.)

III. Node (*) case.

In this case, we see that

- A_+ is homeomorphic to $D^2 \setminus 2D^2$ and
- A_- is the disjoint union of an **Möbius band** and an annulus.

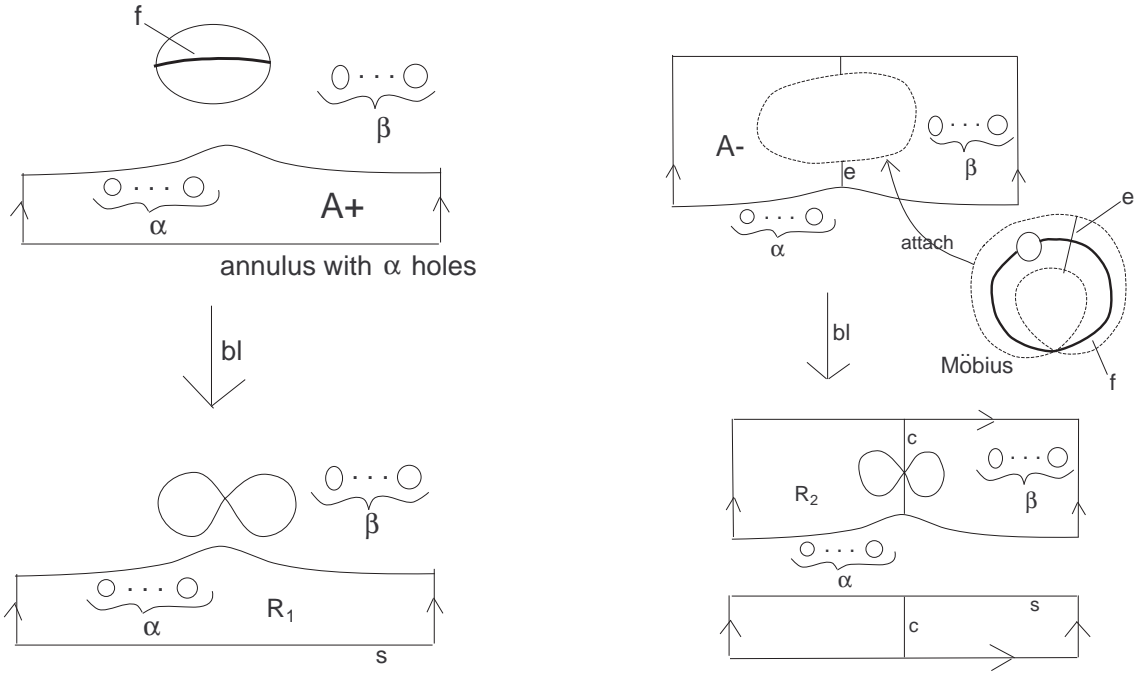


FIGURE 11. The half domains A_+ and A_- . **Node (2)**

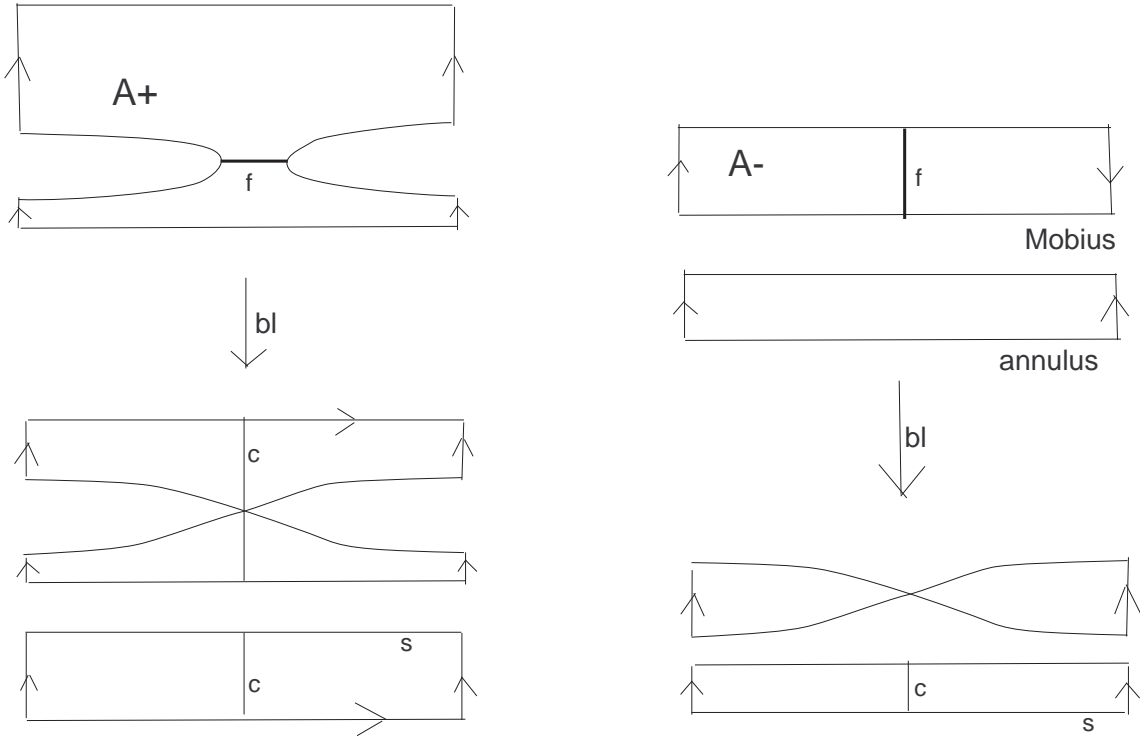


FIGURE 12. The half domains A_+ and A_- . **Node (*)**

See Figure 12.

Suppose that, for φ , $A_- = \pi(X_\varphi(\mathbb{R}))$. Then we see that

$$X_\varphi(\mathbb{R}) \sim T^2 \cup T^2,$$

and

$$A_+ = \pi(X_{\tilde{\varphi}}(\mathbb{R})), \quad X_{\tilde{\varphi}}(\mathbb{R}) \sim \Sigma_2.$$

Moreover, we have

$$H(\varphi) = 0 \quad \text{and} \quad (r(\varphi), a(\varphi), \delta(\varphi)) = (10, 8, 0),$$

and hence $\delta_{\varphi S} = 0$ and $v = 0$.

On the other hand, we have

$$(r(\tilde{\varphi}), a(\tilde{\varphi}), H(\tilde{\varphi})) = (9, 9, \mathbb{Z}/2\mathbb{Z}(F))$$

and $\delta_{\tilde{\varphi} S} = 0$, $v \sim F$.

6.4. Isotopy types of real curves $\mathbb{R}A'_1$ in $|12c + 3s|$ with one real double point on $\mathbb{R}\mathbb{F}_4$.

In Node and Cusp cases, the number of connected components of $\mathbb{R}A'_1$ equals to that of connected components of $\mathbb{R}A_1$. Hence, we have

$$1 \leq \#\{\text{Connected components of } \mathbb{R}A'_1\} \leq 10.$$

In **Node (1)** and **Cusp (1)** cases, we have

$$0 \leq \alpha + \beta \leq 9.$$

If for φ , $A_- = \pi(X_\varphi(\mathbb{R}))$, namely, the invariant $\delta_F(\varphi) = 1$, then $\alpha + \beta = 9 - a(\varphi)$.

In **Node (2)** and **Cusp (2)** cases, we have

$$0 \leq \alpha + \beta \leq 8.$$

If for φ , $A_- = \pi(X_\varphi(\mathbb{R}))$, namely, the invariant $\delta_F(\varphi) = 1$, then $\alpha + \beta = 8 - a(\varphi)$.

In Isolated point case, the number of connected components of $\mathbb{R}A'_1$ equals to that of connected components of $\mathbb{R}A_1$ plus 1.

Hence, we have

$$2 \leq \#\{\text{Connected components of } \mathbb{R}A'_1\} \leq 11.$$

Hence, we have

$$0 \leq \alpha + \beta \leq 9.$$

If for φ , $A_- = \pi(X_\varphi(\mathbb{R}))$, namely, the invariant $\delta_F(\varphi) = 1$, then $\alpha + \beta = 8 - a(\varphi)$.

We already have all the isometry classes (Theorem 1, Table 1 and Table 2) obtained from curves $\mathbb{R}A'_1$ in $|12c + 3s|$ with one double point on $\mathbb{R}\mathbb{F}_4$.

Thus we have:

Theorem 6. *For each isometry class with $H(\varphi) = 0$ ($\Leftrightarrow \delta_F(\varphi) = 1$), we list up all possible isotopy types of curves $\mathbb{R}A'_1$ in $|12c + 3s|$ with one nondegenerate double point on $\mathbb{R}\mathbb{F}_4$ as in Table 4 below, and for each isometry class with $H(\varphi) = \mathbb{Z}/2\mathbb{Z}(F)$ ($\Leftrightarrow \delta_F(\varphi) = 0$), as in Table 5. Note that the isometry class No. k and the isometry class No. k' are related involutions for each $k = 1, \dots, 50$. The isometry class $(10, 8, 0, H(\varphi) = 0)$ and $(9, 9, 0, H(\varphi) = \mathbb{Z}/2\mathbb{Z}(F))$ are also related involutions. Here, we consider only nondegenerate double points. \square*

Isometry class of type $((3, 1, 1), -1)$						Node (1)		Isolated point		Node (2)		Node (*)
No.	r	a	$\delta_{\varphi S}$	g	k	α	β	α	β	α	β	
1	1	1	1	10	0	0	8	0	8	0	7	
2	2	0	0	10	1	1	8	1	8	1	7	
3	2	2	0	9	0	0	7	0	7	0	6	
4	2	2	1	9	0	0	7	0	7	0	6	
5	3	1	1	9	1	1	7	1	7	1	6	
6	3	3	1	8	0	0	6	0	6	0	5	
7	4	2	1	8	1	1	6	1	6	1	5	
8	4	4	1	7	0	0	5	0	5	0	4	
9	5	3	1	7	1	1	5	1	5	1	4	
10	5	5	1	6	0	0	4	0	4	0	3	
11	6	2	0	7	2	2	5	2	5	2	4	
12	6	4	0	6	1	1	4	1	4	1	3	
13	6	4	1	6	1	1	4	1	4	1	3	
14	6	6	1	5	0	0	3	0	3	0	2	
15	7	3	1	6	2	2	4	2	4	2	3	
16	7	5	1	5	1	1	3	1	3	1	2	
17	7	7	1	4	0	0	2	0	2	0	1	
18	8	2	1	6	3	3	4	3	4	3	3	
19	8	4	1	5	2	2	3	2	3	2	2	
20	8	6	1	4	1	1	2	1	2	1	1	
21	8	8	1	3	0	0	1	0	1	0	0	
22	9	1	1	6	4	4	4	4	4	4	3	
23	9	3	1	5	3	3	3	3	3	3	2	
24	9	5	1	4	2	2	2	2	2	2	1	
25	9	7	1	3	1	1	1	1	1	1	0	
26	9	9	1	2	0	0	0	0	0			
27	10	0	0	6	5	5	4	5	4	5	3	
28	10	2	0	5	4	4	3	4	3	4	2	
29	10	2	1	5	4	4	3	4	3	4	2	
30	10	4	0	4	3	3	2	3	2	3	1	
31	10	4	1	4	3	3	2	3	2	3	1	
32	10	6	0	3	2	2	1	2	1	2	0	
33	10	6	1	3	2	2	1	2	1	2	0	
	10	8	0	2	1							$T^2 \cup T^2$
34	10	8	1	2	1	1	0	1	0			
35	11	1	1	5	5	5	3	5	3	5	2	
36	11	3	1	4	4	4	2	4	2	4	1	
37	11	5	1	3	3	3	1	3	1	3	0	
38	11	7	1	2	2	2	0	2	0			
39	12	2	1	4	5	5	2	5	2	5	1	
40	12	4	1	3	4	4	1	4	1	4	0	
41	12	6	1	2	3	3	0	3	0			
42	13	3	1	3	5	5	1	5	1	5	0	
43	13	5	1	2	4	4	0	4	0			
44	14	2	0	3	6	6	1	6	1	6	0	
45	14	4	0	2	5	5	0	5	0			
46	14	4	1	2	5	5	0	5	0			
47	15	3	1	2	6	6	0	6	0			
48	16	2	1	2	7	7	0	7	0			
49	17	1	1	2	8	8	0	8	0			
50	18	0	0	2	9	9	0	9	0			

TABLE 4. Possible isotopy types of curves $\mathbb{R}A'_1$ for each isometry classe of type $((3, 1, 1), -1)$ with $H(\varphi) = 0$

Isometry class of type $((3, 1, 1), -1)$						Node (1)		Isolated point		Node (2)		Node (*)
No.	r	a	$\delta_{\varphi S}$	g	k	α	β	α	β	α	β	
1 ²	18	2	1	1	8	0	8	0	8	0	7	
2 ²	17	1	0	2	8	1	8	1	8	1	7	
3 ²	17	3	0	1	7	0	7	0	7	0	6	
4 ²	17	3	1	1	7	0	7	0	7	0	6	
5 ²	16	2	1	2	7	1	7	1	7	1	6	
6 ²	16	4	1	1	6	0	6	0	6	0	5	
7 ²	15	3	1	2	6	1	6	1	6	1	5	
8 ²	15	5	1	1	5	0	5	0	5	0	4	
9 ²	14	4	1	2	5	1	5	1	5	1	4	
10 ²	14	6	1	1	4	0	4	0	4	0	3	
11 ²	13	3	0	3	5	2	5	2	5	2	4	
12 ²	13	5	0	2	4	1	4	1	4	1	3	
13 ²	13	5	1	2	4	1	4	1	4	1	3	
14 ²	13	7	1	1	3	0	3	0	3	0	2	
15 ²	12	4	1	3	4	2	4	2	4	2	3	
16 ²	12	6	1	2	3	1	3	1	3	1	2	
17 ²	12	8	1	1	2	0	2	0	2	0	1	
18 ²	11	3	1	4	4	3	4	3	4	3	3	
19 ²	11	5	1	3	3	2	3	2	3	2	2	
20 ²	11	7	1	2	2	1	2	1	2	1	1	
21 ²	11	9	1	1	1	0	1	0	1	0	0	
22 ²	10	2	1	5	4	4	4	4	4	4	3	
23 ²	10	4	1	4	3	3	3	3	3	3	2	
24 ²	10	6	1	3	2	2	2	2	2	2	1	
25 ²	10	8	1	2	1	1	1	1	1	1	0	
26 ²	10	10	1	1	0	0	0	0	0			
27 ²	9	1	0	6	4	5	4	5	4	5	3	
28 ²	9	3	0	5	3	4	3	4	3	4	2	
29 ²	9	3	1	5	3	4	3	4	3	4	2	
30 ²	9	5	0	4	2	3	2	3	2	3	1	
31 ²	9	5	1	4	2	3	2	3	2	3	1	
32 ²	9	7	0	3	1	2	1	2	1	2	0	
33 ²	9	7	1	3	1	2	1	2	1	2	0	
	9	9	0	2	0	1	0	1	0			Σ_2
34 ²	9	9	1	2	0	1	0	1	0			
35 ²	8	2	1	6	3	5	3	5	3	5	2	
36 ²	8	4	1	5	2	4	2	4	2	4	1	
37 ²	8	6	1	4	1	3	1	3	1	3	0	
38 ²	8	8	1	3	0	2	0	2	0			
39 ²	7	3	1	6	2	5	2	5	2	5	1	
40 ²	7	5	1	5	1	4	1	4	1	4	0	
41 ²	7	7	1	4	0	3	0	3	0			
42 ²	6	4	1	6	1	5	1	5	1	5	0	
43 ²	6	6	1	5	0	4	0	4	0			
44 ²	5	3	0	7	1	6	1	6	1	6	0	
45 ²	5	5	0	6	0	5	0	5	0			
46 ²	5	5	1	6	0	5	0	5	0			
47 ²	4	4	1	7	0	6	0	6	0			
48 ²	3	3	1	8	0	7	0	7	0			
49 ²	2	2	1	9	0	8	0	8	0			
50 ²	1	1	0	10	0	9	0	9	0			

TABLE 5. Possible isotopy types of curves $\mathbb{R}A'_1$ for each isometry class of type $((3, 1, 1), -1)$ with $H(\varphi) = \mathbb{Z}/2\mathbb{Z}(F)$

For convenience, we distinguish the anti-holomorphic involutions φ_{\pm} on X as follows:

Definition 21 (Definition of φ_{\pm}). *We define the anti-holomorphic involutions φ_{\pm} on X such that*

$$A_{\pm} = \pi(X_{\varphi_{\pm}}(\mathbb{R})).$$

Namely, the fixed point sets of φ_{\pm} doubly-cover the half domains A_{\pm} (Definition 15) respectively.

□

Hence, if $H(\varphi) = 0$ ($\Leftrightarrow \delta_F(\varphi) = 1 \Leftrightarrow [F_{\varphi}(\mathbb{R})] \neq 0$ in $H_1(X_{\varphi}(\mathbb{R}); \mathbb{Z}/2\mathbb{Z})$) for a real 2-elementary K3 surface (X, τ, φ) of type $((3, 1, 1), -1)$, then we have

$$A_{-} = \pi(X_{\varphi}(\mathbb{R}))$$

and

$$\varphi_{-} := \varphi \quad \text{and} \quad \varphi_{+} := \tilde{\varphi} = \tau \circ \varphi.$$

6.5. Next questions. From Tables 4 and 5, we found that an isometry class of involutions of the K3 lattice \mathbb{L}_{K3} of type $((3, 1, 1), -1)$ might contain **several isotopy types** (for example, Node (1), Isolated point and Node (2)) of real curves $\mathbb{R}A'_1$ with one real nondegenerate double point on $\mathbb{R}\mathbb{F}_4$. Equivalently, a connected component of the moduli ([9]) of real 2-elementary K3 surfaces of type $((3, 1, 1), -1)$ might contain several isotopy types of real curves $\mathbb{R}A'_1$. Hence, we next want to classify these isotopy types of real nondegenerate double points.

It is expected that we could be helped by the method of **Itenberg's rigid isotopic classification of real curves of degree 6 on \mathbb{RP}^2 with one nondegenerate double point** ([3],[4],[5]). This classification corresponds to that of real nonsingular curves A in $|-2K_{\mathbb{F}_1}|$ on $\mathbb{R}\mathbb{F}_1$ when we blow up \mathbb{P}^2 at the nondegenerate double point to the Hirzebruch surface \mathbb{F}_1 . The double coverings X of \mathbb{F}_1 ramified along the nonsingular curves A are real 2-elementary K3 surfaces of type $(\langle 2 \rangle \oplus \langle -2 \rangle, -1)$.

Moreover, the classification of real curves of degree 6 on \mathbb{RP}^2 with one nondegenerate double point is related to the “**non-increasing simplest degenerations**” (conjunctions and contractions) of real **nonsingular** curves of degree 6 on \mathbb{RP}^2 . Hence, it is expected that we should pay attention to the degenerations of real nonsingular curves in $|-2K_{\mathbb{F}_4}|$ on $\mathbb{R}\mathbb{F}_4$.

We are also interested in the correspondence of the graphs describing degenerations of nonsingular curves and the Coxeter graphs ([11], [3], [4]) which are obtained from isometry classes of involutions of the K3 lattice \mathbb{L}_{K3} of type $((3, 1, 1), -1)$.

7. REAL CURVES OF DEGREE 6 WITH ONE NONDEGENERATE DOUBLE POINT – ITENBERG'S ARGUMENTS –

In this section we **review** Itenberg's methods and arguments ([3], [4]).

7.1. Real curve of degree 6 on \mathbb{P}^2 with one real nondegenerate double point. Let A be a real curve of degree 6 on \mathbb{P}^2 with one real nondegenerate double point. We blow up \mathbb{P}^2 at the nondegenerate double point to the Hirzebruch surface \mathbb{F}_1 . We get the proper transform (nonsingular curve) $A' (\subset \mathbb{F}_1)$ of A . A' is contained in $|-2K_{\mathbb{F}_1}|$. Then the double covering

$$X$$

of \mathbb{F}_1 ramified along A' is a K3 surfaces. Let $\tau : X \rightarrow X$ be its covering transformation.

The complex conjugation on P^2 is lifted into the anti-holomorphic involution on \mathbb{F}_1 . Moreover, it is lifted into two anti-holomorphic involutions φ and φ' , where $\varphi' = \tau \circ \varphi$, on the K3 surface X . Let h be the preimage ($\subset X$) of the total transform in \mathbb{F}_1 of $\mathbb{P}^1 (\subset \mathbb{P}^2)$. Let δ be the exceptional section of \mathbb{F}_1 . Then we have

$$h^2 = 2, \quad \delta^2 = -2, \quad \text{and} \quad h \cdot \delta = 0.$$

The elements h and δ generate the fixed part of τ_* , i.e., $H_{2+}(X; \mathbb{Z})$. Obviously we have $\varphi_*(h) = -h$, $\varphi'_*(h) = -h$, $\varphi_*(\delta) = -\delta$, $\varphi'_*(\delta) = -\delta$. Thus, (X, τ, φ) and (X, τ, φ') are real 2-elementary K3 surfaces of type

$$(\langle 2 \rangle \oplus \langle -2 \rangle, -1).$$

Let $\mathbb{R}A$ be the real part of A . $\mathbb{R}A$ divide $\mathbb{R}P^2$ into two half domains $\mathbb{R}P_{inner}^2$ and $\mathbb{R}P_{outer}^2$, which could be $\{f \geq 0\}$ and $\{f \leq 0\}$ respectively, where f is a defining real homogenous sextic polynomial of the curve A .

Convention: If one of two sets is **non-orientable**, then we can take $\mathbb{R}P_{outer}^2$ **non-orientable**, which contains so-called the “outermost” component.

Definition 22. We define φ and φ' such that the fixed point set $X_\varphi(\mathbb{R})$ of φ covers $\mathbb{R}P_{outer}^2$ and $X_{\varphi'}(\mathbb{R})$ covers $\mathbb{R}P_{inner}^2$. \square

7.2. The period domain and its connected components. We set $(S, \theta) := (\langle 2 \rangle \oplus \langle -2 \rangle, -1)$. Let h', δ' be generators of lattices $\langle 2 \rangle, \langle -2 \rangle$ respectively. Note that we have the group $G = \{1_S\}$ for $(S, \theta) := (\langle 2 \rangle \oplus \langle -2 \rangle, -1)$.

We describe an outline of Itenberg [3]’s argument as below.

Fix an involution (\mathbb{L}_{K3}, ψ) of type $(\langle 2 \rangle \oplus \langle -2 \rangle, -1)$ and use the notation

$$\Omega / -\psi = \mathcal{L}_+ \times \mathcal{L}_{-,S},$$

e.t.c. defined in Subsections 2.1 — 2.3.

Lemma 7 (Itenberg [3], p.281, a criterion). *For $[\omega] \in \Omega / -\psi$, $[\omega]$ is the period of a marked real 2-elementary K3 surface obtained from a real curve of degree 6 on $\mathbb{R}P^2$ with one nondegenerate double point. if and only if there are no \mathbf{v} ($\neq \pm\delta'$) ($\in \mathbb{L}_{K3}$) satisfying that $\mathbf{v} \cdot \omega = 0$, $\mathbf{v} \cdot h' = 0$ and $\mathbf{v}^2 = -2$.* \square

(-2)-orthogonal hyperplanes and reflections against them. The reflection on \mathcal{L}_+ against the real hyperplane \mathbf{v}^\perp where \mathbf{v} is an element of \mathbb{L}_+ with square -2 is welldefined and it sends a point in $\Omega / -\psi$ to its equivalent point. So is the reflection on $\mathcal{L}_{-,S}$ against the real hyperplane \mathbf{v}^\perp where \mathbf{v} is an element of $\mathbb{L}_{-,S}$ with square -2 . Hence, let

$$\Omega_+ \quad (\text{respectively, } \Omega_{-,S})$$

be the **open fundamental domains** with respect to the groups generated by the reflections against the real hyperplanes \mathbf{v}^\perp satisfying $\mathbf{v}^2 = -2$ and $\mathbf{v} \in \mathbb{L}_+$ (respectively, $\mathbb{L}_{-,S}$) and we consider the direct product

$$\Omega_+ \times \Omega_{-,S}.$$

The periods of marked real 2-elementary K3 surfaces $(X, \tau, \varphi, \alpha)$ obtained from some real curves of degree 6 with one nondegenerate double point and satisfying $\alpha\varphi_*\alpha^{-1} = \psi$ (see above) **belong to** $\Omega_+ \times \Omega_{-,S}$ up to equivalence.

(-6)-orthogonal hyperplanes. We have removed the real hyperplanes \mathbf{v}^\perp such that **either** $\mathbf{v} \in \mathbb{L}_+$ **or** $\mathbf{v} \in \mathbb{L}_{-,S}$. Moreover, we have to remove more $\omega = \omega_+ + \omega_-$ orthogonal to some \mathbf{v} such that \mathbf{v} ($\neq \pm\delta'$) ($\in \mathbb{L}_{K3}$), $\mathbf{v} \cdot h' = 0$, $\mathbf{v}^2 = -2$, $\mathbf{v} \notin \mathbb{L}_+$ and $\mathbf{v} \notin \mathbb{L}_{-,S}$.

Let $\mathbf{v} = \mathbf{v}_+ + \mathbf{v}_-$ where $\mathbf{v}_+ \in \mathbb{L}_+ \otimes \mathbb{Q}$, $\mathbf{v}_- \in \mathbb{L}_- \otimes \mathbb{Q}$. We set

$$\mathbb{L}_{-,h'} := \mathbb{L}_- \cap (h')^\perp.$$

Since $2\mathbf{v}_- \in \mathbb{L}_-$, we have $2\mathbf{v}_- \in \mathbb{L}_{-,h'}$. If $\omega = \omega_+ + \omega_- \in \Omega_+ \times \Omega_{-,S}$, then $\mathbf{v}_- \neq 0$. In fact, if $\mathbf{v}_- = 0$, then $\mathbf{v} = \mathbf{v}_+ \in \mathbb{L}_+$ and such ω has already been removed. Now we consider two cases.

a) $\mathbf{v}_+ = 0$. If $\mathbf{v} = \mathbf{v}_-$ does not belong to $\mathbb{L}_{-,S}$, then let \mathbf{v}'_- be the projection of \mathbf{v}_- on $\mathbb{L}_{-,S} \otimes \mathbb{Q}$. Since $\mathbb{L}/(S \oplus S^\perp)$ is 2-elementary, we have $2\mathbf{v}'_- \in \mathbb{L}_{-,S}$, and $\mathbf{v} = \mathbf{v}_- = \mathbf{v}'_- + (\pm 1)/2 \delta'$. Hence we have

$$(2\mathbf{v}'_-)^2 = -6 \quad \text{and} \quad 2\mathbf{v}'_- \equiv \delta' \pmod{2\mathbb{L}_{K3}}.$$

Conversely, suppose that $\omega_- \cdot 2\mathbf{v}'_- = 0$ for some $2\mathbf{v}'_- \in \mathbb{L}_{-,S}$ with

$$(2\mathbf{v}'_-)^2 = -6 \quad \text{and} \quad 2\mathbf{v}'_- \equiv \delta' \pmod{2\mathbb{L}_{K3}}.$$

Then, for $\omega = \omega_+ + \omega_- \in \Omega_+ \times \Omega_{-,S}$, we have

$$\omega \cdot \mathbf{v}_- = (\omega_+ + \omega_-) \cdot (\mathbf{v}'_- + (1/2)\delta') = \omega_- \cdot (\mathbf{v}'_- + (1/2)\delta') = \omega_- \cdot \mathbf{v}'_- = 0.$$

Therefore, we have to remove such ω from $\Omega_+ \times \Omega_{-,S}$. So, we remove some **(-6)-orthogonal real hyperplanes** from $\Omega_{-,S}$, and we get a collection of open polytopes

$$\Omega_{-,S}^i \quad (i = 1, 2, \dots).$$

b) $\mathbf{v}_+ \neq 0$. We have $-2 = (\mathbf{v})^2 = (\mathbf{v}_+)^2 + (\mathbf{v}_-)^2 = (\mathbf{v}_+ - \mathbf{v}_-)^2$. If $\mathbf{v}_+ + \mathbf{v}_-$, $\mathbf{v}_+ - \mathbf{v}_-$, δ' are linearly dependent, then we can easily check that

$$(2\mathbf{v}_+)^2 = -6 \quad \text{and} \quad 2\mathbf{v}_+ \equiv \delta' \pmod{2\mathbb{L}_{K3}}.$$

Of course, $2\mathbf{v}_+ \in \mathbb{L}_+$. We have to remove $\omega = \omega_+ + \omega_-$ satisfying $\omega_+ \cdot 2\mathbf{v}_+ = 0$ for $2\mathbf{v}_+ \in \mathbb{L}_+$ as above. So, we remove some **(-6)-orthogonal real hyperplanes** from Ω_+ , and we get a collection of open polytopes

$$\Omega_+^j \quad (j = 1, 2, \dots).$$

The real linear subspace Z . If $\mathbf{v}_+ + \mathbf{v}_-$, $\mathbf{v}_+ - \mathbf{v}_-$, δ' are linearly independent, then we have to remove an appropriate real linear subspace Z of **real codimension ≥ 2** . Finally, we get

Theorem 8 (Itenberg [3], p.282). *The image of our period map (the period domain) is*

$$\left(\bigcup_{i,j} \Omega_+^j \times \Omega_{-,S}^i \right) \setminus Z.$$

Remark that **The number of connected components of $\bigcup_{i,j} \Omega_+^j \times \Omega_{-,S}^i$ does not change** after we remove the set Z from it.

Moreover, each class $[\omega]$ corresponds to a **unique** (up to projective equivalence) marked real 2-elementary K3 surface obtained from a real curve of degree 6 with one nondegenerate double point.

Remark 16. For (\mathbb{L}_{K3}, ψ) , either \mathbb{L}_+ or $\mathbb{L}_{-,S}$ does not contain any element \mathbf{v} such that $\mathbf{v} \equiv \delta' \pmod{2\mathbb{L}_{K3}}$. Hence, we have the following.

If $\mathbb{L}_{-,S}$ contains an element \mathbf{v} such that $\mathbf{v} \equiv \delta' \pmod{2\mathbb{L}_{K3}}$, then \mathbb{L}_+ does not contain any \mathbf{v} such that $\mathbf{v} \equiv \delta' \pmod{2\mathbb{L}_{K3}}$, and hence, $\bigcup_j \Omega_+^j$ consists of a **single polytope**.

If $\mathbb{L}_{-,S}$ does not contain any \mathbf{v} such that $\mathbf{v} \equiv \delta' \pmod{2\mathbb{L}_{K3}}$, then $\bigcup_i \Omega_{-,S}^i$ consists of a **single polytope**. \square

Definition 23 (Itenberg [3], p.282, Definition of $\Omega_*^{\mathbb{R}P^2}$).

$$\Omega_*^{\mathbb{R}P^2} := \begin{cases} \bigcup_i \Omega_{-,S}^i & \text{if } \mathbb{L}_{-,S} \text{ contains an element } \mathbf{v} \text{ such that } \mathbf{v} \equiv \delta' \pmod{2\mathbb{L}_{K3}}, \\ \bigcup_j \Omega_+^j & \text{if } \mathbb{L}_{-,S} \text{ does not contain any element } \mathbf{v} \text{ such that } \mathbf{v} \equiv \delta' \pmod{2\mathbb{L}_{K3}}. \end{cases}$$

Then, we have:

Theorem 9 (Itenberg [3], Theorem 2.1). *We fix ⁴ an involution (\mathbb{L}_{K3}, ψ) of type $(\langle 2 \rangle \oplus \langle -2 \rangle, -1)$. Rigid isotopy classes (up to projective equivalence) of real curves of degree 6 with one nondegenerate double point which yield marked real 2-elementary K3 surfaces $((X, \tau, \varphi), \alpha)$ satisfy $\alpha \varphi_* \alpha^{-1} = \psi$ are*

⁴ Recall Remark 7.

in one to one (bijective) correspondence with the connected components (“polytopes”) (considered up to equivalence) of $\Omega_*^{\mathbb{R}P^2}$. \square

Remark 17. Here we can **choose** (see Remark 16) an anti-holomorphic involution (φ or φ') on the $K3$ surface X such that $\mathbb{L}_{-,S}$ contains an element \mathbf{v} such that $\mathbf{v} \equiv \delta' \pmod{2\mathbb{L}_{K3}}$. Hence, we may assume that (see Definition 23)

$$\Omega_*^{\mathbb{R}P^2} = \bigcup_i \Omega_{-,S}^i.$$

7.3. The Coxeter graph C and the graph K . We are fixing an involution (\mathbb{L}_{K3}, ψ) of type $(\langle 2 \rangle \oplus \langle -2 \rangle, -1)$ and the elements h', δ' are generators of $\langle 2 \rangle, \langle -2 \rangle$ respectively. Let

$$\mathcal{L}_{-,h'}$$

be the Lobachevsky space obtained from $\mathbb{L}_{-,h'} \otimes \mathbb{R}$. (Here, the element h' corresponds to the preimage ($\subset X$) of the total transform in \mathbb{F}_1 of $\mathbb{P}^1 (\subset \mathbb{P}^2)$, and $(h')^2 = 2$.)

The group generated by the reflections against real hyperplanes \mathbf{v}^\perp such that $\mathbf{v} \in \mathbb{L}_{-,h'}$ with $\mathbf{v}^2 = -2$ acts on $\mathcal{L}_{-,h'}$. Let

$$\widetilde{\Omega}_-$$

be its fundamental domain **having a face which is orthogonal to δ'** , and let

$$C$$

be the **Coxeter graph** (see Vinberg [11]) of $\widetilde{\Omega}_-$.

Definition 24 (Itenberg [3], p.283). *We define the graph K as follows.*

- *Let C' be the graph obtained from C by removing all thick or dotted edges.* ⁵
- *Consider the group which consists of symmetries of C' obtained from some automorphisms of $(\mathbb{L}_{K3}, \psi, h')$. Let C'' be the graph which is the quotient of C' by the group.*
- *Let e be the vertex in C which corresponds to δ' , and e' be the class in C'' containing e .*
- *Let K be the connected component of C'' containing e' .* \square

Theorem 10 (Itenberg [3], Proposition 3.1). *The number of polytopes (up to equivalence) in $\Omega_*^{\mathbb{R}P^2}$ coincides with that of vertices of the graph K .* \square

7.4. Degenerations of nonsingular real curves of degree 6, the graph P and P' . We next describe the rigid isotopy classes in Theorem 9 from the point of view of **degenerations of nonsingular sextic curves**.

Let C_0 be a real curve of degree 6 with one nondegenerate double point. For such a curve C_0 , we consider a smoothing C_t ($t \in \mathbb{R}$) such that

$$\#\{\text{ovals of nonsingular curves } C_{t_{-1}}\} \geq \#\{\text{ovals of nonsingular curves } C_{t_1}\}$$

for any $t_{-1} < 0$ and any $0 < t_1$. We call ([3]) a family C_t ($t_{-1} \leq t \leq 0$) **non-increasing simplest degeneration** of $C_{t_{-1}}$ to C_0 . ⁶

We regard a real curves of degree 6 with one nondegenerate double point as the results of some **non-increasing simplest degeneration** of a real nonsingular curves of degree 6 as follows.

- **Conjunction 1)** The conjunction of the outermost oval and an empty oval in **the exterior** of the outermost oval.
- **Conjunction 1')** The conjunction of the outermost oval and an empty oval in **the interior** of the outermost oval.
- **Conjunction 2)** The conjunction of two empty ovals in **the exterior** of the outermost oval.

⁵Two vertices are connected by thick or dotted edges if their value of the bilinear form ≥ 2 .

⁶A family C_t ($t_1 \geq t \geq 0$) is called **non-decreasing simplest degeneration** of C_{t_1} to C_0 .

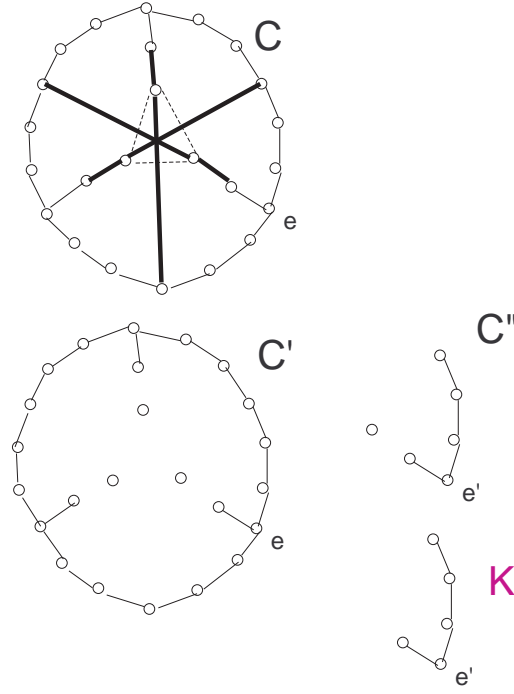


FIGURE 13. An example of the graphs K . ([3])

- **Conjunction 2')** The conjunction of two empty ovals in **the interior** of the outermost oval.
- **Contraction 3)** The contraction of an empty oval in **the exterior** of the outermost oval.
- **Contraction 3')** The contraction of an empty oval in **the interior** of the outermost oval.
- **Conjunction 4)** The conjunction of the outermost oval with itself in such a way that the obtained curve is embedded in $\mathbb{R}P^2$ like the union of two real lines.
- **Conjunction 5)** The conjunction of an empty oval in the exterior of the outermost oval with itself in such a way that the obtained curve is embedded in $\mathbb{R}P^2$ like the union of two real lines.

Lemma 11 (Itenberg [3], Proposition 3.3). **Fix** a real nonsingular curve A' of degree 6.

- For real curves of degree 6 with one nondegenerate double point being the results of degenerations of types 1)—3) of the real nonsingular curve A' of degree 6, we should **choose** “ φ ” (see Definition 22 and Remark 17) as the anti-holomorphic involutions on X . And for all marked real 2-elementary $K3$ surfaces $((X, \tau, \varphi), \alpha)$, the associated involutions of \mathbb{L}_{K3} are **isometric**, namely, all such (X, τ, φ) have their markings to the same involution of \mathbb{L}_{K3} .
- For real curves of degree 6 with one nondegenerate double point being the results of degenerations of types 1')—3') of the real nonsingular curve A' of degree 6, we should **choose** “ φ' ” (see Definition 22 and Remark 17) as the anti-holomorphic involutions on X . And for all marked real 2-elementary $K3$ surfaces (X, τ, φ') , the associated involutions $(H_2(X, \mathbb{Z}), (\varphi')_*)$ are **isometric**, namely, all such (X, τ, φ') have their markings to the same involution of \mathbb{L}_{K3} . \square

Definition 25 (Itenberg [3], The graphs P and P'). We fix a nonsingular real curve A' of degree 6 and define the graph P as follows.

- The **vertices** of P are the **rigid isotopy classes** of real curves of degree 6 with one nondegenerate double point being the results of degenerations of types 1)—3) of A' .
- Two vertices of P are connected by a **edge** if one rigid isotopy class is obtained from the conjunction of ovals \mathcal{E}_1 and \mathcal{E}_2 of A' , and the other rigid isotopy class is obtained from the contraction of the oval \mathcal{E}_1 .

We define the graph P'^7 similarly. \square

Then we have

Theorem 12 (Itenberg [3], Proposition 3.4). *Fix a nonsingular real curve A' of degree 6. Get a real curves of degree 6 with one nondegenerate double point being the result of one of degenerations of types 1)–3) of the nonsingular curve A' , and construct the graph K (Definition 24). Then, K is isomorphic to P .*⁸ \square

8. DEGENERATIONS OF NONSINGULAR REAL CURVES IN $|12c + 3s|$ ON $\mathbb{R}\mathbb{F}_4$

8.1. Review of nonsingular real curves in $|-2K_{\mathbb{F}_4}|$ on $\mathbb{R}\mathbb{F}_4$. This subsection is a review of the result of the last section of [9].

We consider the case

$$S \cong \mathbb{U} \text{ and } \theta = -1.$$

In this case $Y := X/\{1, \tau\} = \mathbb{F}_4$, and all real 2-elementary K3 surfaces are **(\mathcal{D})-nondegenerate**. The group G is trivial. Since \mathbb{U} is unimodular, we have $H(\varphi) = 0$ (cf. Section 5) and

$$\delta_{\varphi S} = \delta_{\varphi}.$$

In this case the genus defines the isometry class of an involution $(\mathbb{L}_{K3}, \varphi)$ of type $(\mathbb{U}, -1)$ and it is determined by the data

$$(8.1) \quad (r, a, \delta_{\varphi S} = \delta_{\varphi}, v)$$

where the “characteristic element” $v = 0$ if $\delta_{\varphi S} = 0$ (otherwise, v is not defined). The complete list of the data (8.1) is given in Figure 14. There are 14 isometry classes with $\delta_{\varphi} = 0$ and 49 isometry classes with $\delta_{\varphi} = 1$. Thus we have 63 classes.

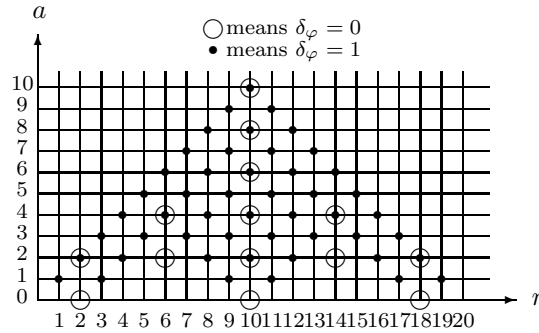


FIGURE 14. \mathbb{F}_4 : All possible data (r, a, δ_{φ}) . (**always** $H(\varphi) = 0$)

For “related involutions” ([9]), we have

$$r(\varphi) + r(\tilde{\varphi}) = 20, \quad a(\varphi) = a(\tilde{\varphi}) \quad \text{and} \quad \delta_{\varphi S} = \delta_{\tilde{\varphi} S}, \quad s_{\varphi} = s_{\tilde{\varphi}}.$$

If we identify related involutions, there are 10 isometry classes with $\delta_{\varphi} = 0$ and 27 isometry classes with $\delta_{\varphi} = 1$. Thus we have 37 classes.

Let us consider the geometric interpretation of the above results.

⁷ The **vertices** of P' are the **rigid isotopy classes** of real curves of degree 6 with one nondegenerate double point being the results of degenerations of types 1')–3') of A' . Two vertices of P are connected by a **edge** if one rigid isotopy class is obtained from the conjunction of ovals \mathcal{E}_1 and \mathcal{E}_2 of the nonsingular curve A' , and the other rigid isotopy class is obtained from the contraction of the oval \mathcal{E}_1 .

⁸ In the same way, get a real curves of degree 6 with one nondegenerate double point being the result of one of degenerations of types 1')–3') of the nonsingular curve A' , and construct the graph K (Definition 24). Then, K is isomorphic to P' .

Denote by s the exceptional rational section with $s^2 = -4$ of \mathbb{F}_4 , and by c the fiber of the natural fibration $f : \mathbb{F}_4 \rightarrow s$. One has $c^2 = 0$. We have $-2K_{\mathbb{F}_4} = 12c + 4s$. Thus, for $A \in |-2K_{\mathbb{F}_4}|$, one has $A \cdot s = -4$. It follows

$$A = s + A_1$$

where $A_1 \in |12c + 3s|$. We have $c \cdot A_1 = 3$ and $s \cdot A_1 = 0$.

It follows that a **nonsingular** $A \in |-2K_{\mathbb{F}_4}|$ **has two irreducible components** s and A_1 . We describe the connected components of moduli of nonsingular curves $A_1 \in |12c + 3s|$.

We mention that the lattice $S \cong \mathbb{U} \cong \mathbb{Z}C + \mathbb{Z}E$ where $C = \pi^*(c)$ and $E = \pi^*(s)/2$. We have $C^2 = 0$, $E^2 = -2$ and $C \cdot E = 1$. Here, we denote by $\pi : X \rightarrow Y = \mathbb{F}_4$ the quotient map.

If $Y = \mathbb{R}\mathbb{F}_4$ is not empty, then $\mathbb{R}\mathbb{F}_4$ is a **torus** and $\mathbb{R}s$ is a circle giving a generator of the torus. The curves $\mathbb{R}c$ where $f(c) \in \mathbb{R}s$ give another generator of the torus. It follows that *a real curve $\mathbb{R}A_1$ belongs to the open cylinder $\mathbb{R}\mathbb{F}_4 - \mathbb{R}s$ with its infinity identified with two copies of $\mathbb{R}s$.*

Using $c \cdot A_1 = 3$, for both involutions φ and $\tau \circ \varphi$, we get that a half domain ($=$ “positive curve”) $A_- = \pi(X_\varphi(\mathbb{R}))$ with the invariants (8.1) has the isotopy type given in Figure 15.

In Figure 15 we assume that $(r, a, \delta_\varphi) \neq (10, 10, 0)$. Then $\mathbb{R}\mathbb{F}_4$ is a torus, and

$$g = (22 - r - a)/2 \quad \text{and} \quad k = (r - a)/2.$$

If $(r, a, \delta_\varphi) = (10, 10, 0)$, namely, $(r, a, \delta_{\varphi S}, v) = (10, 10, 0, 0)$, then $\mathbb{R}\mathbb{F}_4 = \emptyset$.

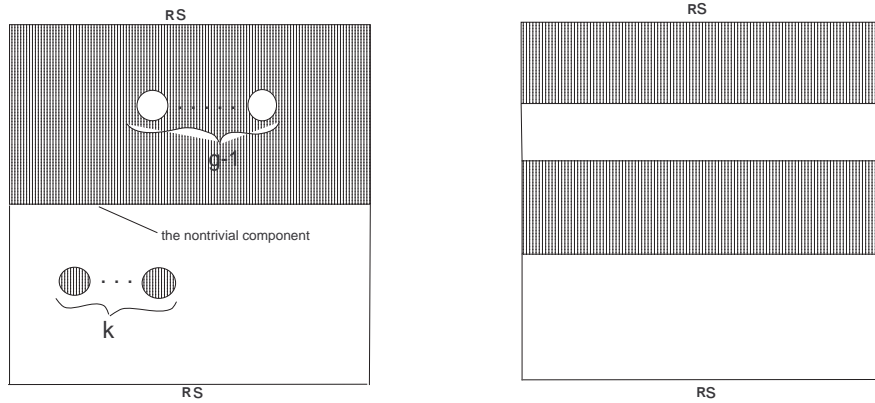


FIGURE 15. \mathbb{F}_4 : A_- with $(r, a, \delta_{\varphi S}, v) \neq (10, 8, 0, 0), (10, 10, 0, 0)$ and A_- with $(r, a, \delta_{\varphi S}, v) = (10, 8, 0, 0)$

Theorem 13 (Nikulin and Saito [9]). *A connected component of the moduli of the half domains ($=$ real nonsingular “positive curves”) $A_- = \pi(X_\varphi(\mathbb{R}))$, $A \in |-2K_{\mathbb{F}_4}|$, up to the action of the automorphism group of \mathbb{F}_4 over \mathbb{R} is determined by the isotopy type and the invariant $\delta_\varphi = \delta_{\varphi S}$. All possible data are presented in Figure 14. See also Figure 15. \square*

8.2. Degenerations of nonsingular real curves in $|12c + 3s|$ on $\mathbb{R}\mathbb{F}_4$. We next consider “non-increasing simplest degenerations” (conjunctions, contractions) of **nonsingular** real curves in

$$|-2K_{\mathbb{F}_4}| = |12c + 4s|$$

on $\mathbb{R}\mathbb{F}_4$, which are **analogies of** Subsection 7.4.

As stated in the previous subsection, a nonsingular curve in $|-2K_{\mathbb{F}_4}|$ has two irreducible components

$$s \text{ and } A_1,$$

where A_1 is a nonsingular curve in

$$|12c + 3s|.$$

Let C_0 be real curve in $|12c + 3s|$ on \mathbb{F}_4 with one nondegenerate double point. For such a curve C_0 , we consider a smoothing C_t ($t \in \mathbb{R}$) such that

$$\#\{\text{ovals of nonsingular curves } C_{t_{-1}}\} \geq \#\{\text{ovals of nonsingular curves } C_{t_1}\}$$

for any $t_{-1} < 0$ and any $0 < t_1$.

We call a family C_t ($t_{-1} \leq t \leq 0$) **non-increasing simplest degeneration** of $C_{t_{-1}}$ to C_0 .

We want to regard a real curve $\mathbb{R}A'_1$ in $|12c + 3s|$ on \mathbb{F}_4 with **one nondegenerate double point** as the result of some **non-increasing simplest degeneration** of a real **nonsingular** curve in $|12c + 3s|$ on \mathbb{F}_4 as follows.

Definition 26 (Degenerations 1)—3), 1')—3'), 4) and 5)). *First we fix each isometry class of integral involutions of type $(\mathbb{U}, -1)$ with $(r, a, \delta_{\varphi S}, v) \neq (10, 8, 0, 0), (10, 10, 0, 0)$. Let us consider a corresponding real 2-elementary K3 surfaces (X, τ, φ) of type $(\mathbb{U}, -1)$. Let $\pi : X \rightarrow X/\tau = \mathbb{F}_4$ be the quotient map (double covering). Then we have a real **nonsingular** curve in $|12c + 3s|$ on $\mathbb{R}\mathbb{F}_4$. Suppose that the fixed point set $X_\varphi(\mathbb{R})$ is homeomorphic to*

$$\Sigma_g \cup kS^2.$$

Then the half domain $\pi(X_\varphi(\mathbb{R}))$ ($\subset \mathbb{R}\mathbb{F}_4$) is the disjoint union of an annulus with $(g-1)$ holes and k disks. See Figure 15. The boundary of the annulus with $(g-1)$ holes consists of one nontrivial component, $\mathbb{R}s$ and $(g-1)$ empty ovals.

- **Conjunction 1)** *The conjunction of the nontrivial component and one of the $(g-1)$ empty ovals. Here, remark that the domain surrounded by the nontrivial component, $\mathbb{R}s$ and the $(g-1)$ empty ovals is covered by the fixed point set of the anti-holomorphic involution φ .*
- **Conjunction 1')** *The conjunction of the nontrivial component and one of the k empty ovals. Here, remark that the domain surrounded by the nontrivial component, $\mathbb{R}s$ and the k empty ovals is covered by the fixed point set of the related involution $\tilde{\varphi}$ of the anti-holomorphic involution φ .*
- **Conjunction 2)** *The conjunction of two of the $(g-1)$ empty ovals. Here, remark that the domain surrounded by the nontrivial component, $\mathbb{R}s$ and the $(g-1)$ empty ovals is covered by the fixed point set of the anti-holomorphic involution φ .*
- **Conjunction 2')** *The conjunction of two of the k empty ovals. Here, remark that the domain surrounded by the nontrivial component, $\mathbb{R}s$ and the k empty ovals is covered by the fixed point set of the related involution $\tilde{\varphi}$ of the anti-holomorphic involution φ .*
- **Contraction 3)** *The contraction of one of the $(g-1)$ empty ovals. Here, remark that the domain surrounded by the nontrivial component, $\mathbb{R}s$ and the $(g-1)$ empty ovals is covered by the fixed point set of the anti-holomorphic involution φ .*
- **Contraction 3')** *The contraction of one of the k empty ovals. Here, remark that the domain surrounded by the nontrivial component, $\mathbb{R}s$ and the k empty ovals is covered by the fixed point set of the related involution $\tilde{\varphi}$ of the anti-holomorphic involution φ .*

We next consider **the isometry class** of integral involutions of type $(\mathbb{U}, -1)$ with $(r, a, \delta_{\varphi S}, v) = (10, 8, 0, 0)$.

- **Conjunction 4)** *The two nontrivial components conjunct and become the union of two real lines (Node (*)) in $\mathbb{R}\mathbb{F}_4$.*

We finally consider the isometry class of integral involutions of type $(\mathbb{U}, -1)$ with $(r, a, \delta_{\varphi S}, v) = (9, 9, 1)$ or $(11, 9, 1)$.

- **Conjunction 5)** *An empty oval conjuncts with itself and becomes the union of two real lines (Node (*)) in $\mathbb{R}\mathbb{F}_4$. \square*

See Figure 16. Compare with Figure 15.

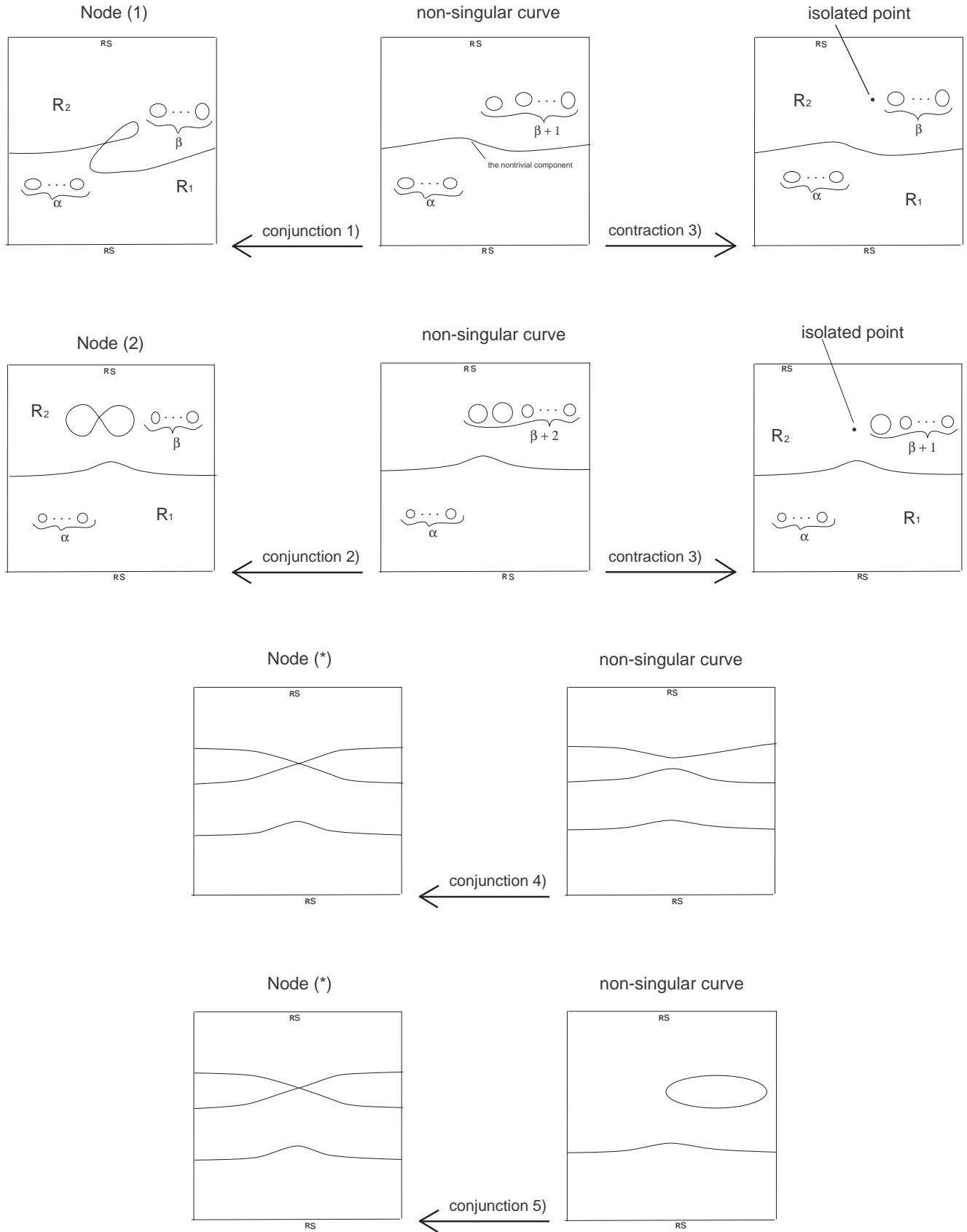


FIGURE 16. **Nonsingular** real curves in $| -2K_{\mathbb{F}_4} |$ on $\mathbb{R}\mathbb{F}_4$ and their degenerations.

Then we have:

Theorem 14. *We list up all possible non-increasing simplest degenerations in Tables 6 — 8 for each isometry class of 63 involutions of type $(\mathbb{U}, -1)$ with the invariant $(r, a, \delta_\varphi, v)$, equivalently, each pair of a real nonsingular curve in $|-2K_{\mathbb{F}_4}|$ on \mathbb{F}_4 and an anti-holomorphic involution φ on X . \square*

isometry class of type $(\mathbb{U}, -1)$ (nonsingular curve)						conjunction 1) conjunction 1')	conjunction 2) conjunction 2')	contraction 3) contraction 3')
r	a	δ_φ	g	k	g - 1	\rightarrow Node (1)	\rightarrow Node (2)	\rightarrow Isolated point
1	1	1	10	0	9	α, β	α, β	α, β
1	1	1	10	0	9	0, 8	0, 7	0, 8
1	1	1	10	0	9	impossible	impossible	impossible
2	0	0	10	1	9	1, 8	1, 7	1, 8
2	0	0	10	1	9	9, 0	impossible	9, 0
2	2	0	9	0	8	0, 7	0, 6	0, 7
2	2	0	9	0	8	impossible	impossible	impossible
2	2	1	9	0	8	0, 7	0, 6	0, 7
2	2	1	9	0	8	impossible	impossible	impossible
3	1	1	9	1	8	1, 7	1, 6	1, 7
3	1	1	9	1	8	8, 0	impossible	8, 0
3	3	1	8	0	7	0, 6	0, 5	0, 6
3	3	1	8	0	7	impossible	impossible	impossible
4	2	1	8	1	7	1, 6	1, 5	1, 6
4	2	1	8	1	7	7, 0	impossible	7, 0
4	4	1	7	0	6	0, 5	0, 4	0, 5
4	4	1	7	0	6	impossible	impossible	impossible
5	3	1	7	1	6	1, 5	1, 4	1, 5
5	3	1	7	1	6	6, 0	impossible	6, 0
5	5	1	6	0	5	0, 4	0, 3	0, 4
5	5	1	6	0	5	impossible	impossible	impossible
6	2	0	7	2	6	2, 5	2, 4	2, 5
6	2	0	7	2	6	6, 1	6, 0	6, 1
6	4	0	6	1	5	1, 4	1, 3	1, 4
6	4	0	6	1	5	5, 0	impossible	5, 0
6	4	1	6	1	5	1, 4	1, 3	1, 4
6	4	1	6	1	5	5, 0	impossible	5, 0
6	6	1	5	0	4	0, 3	0, 2	0, 3
6	6	1	5	0	4	impossible	impossible	impossible
7	3	1	6	2	5	2, 4	2, 3	2, 4
7	3	1	6	2	5	5, 1	5, 0	5, 1
7	5	1	5	1	4	1, 3	1, 2	1, 3
7	5	1	5	1	4	4, 0	impossible	4, 0
7	7	1	4	0	3	0, 2	0, 1	0, 2
7	7	1	4	0	3	impossible	impossible	impossible
8	2	1	6	3	5	3, 4	3, 3	3, 4
8	2	1	6	3	5	5, 2	5, 1	5, 2
8	4	1	5	2	4	2, 3	2, 2	2, 3
8	4	1	5	2	4	4, 1	4, 0	4, 1
8	6	1	4	1	3	1, 2	1, 1	1, 2
8	6	1	4	1	3	3, 0	impossible	3, 0
8	8	1	3	0	2	0, 1	0, 0	0, 1
8	8	1	3	0	2	impossible	impossible	impossible

TABLE 6. Nonsingular real curves in $|-2K_{\mathbb{F}_4}|$ on $\mathbb{R}\mathbb{F}_4$ and their possible degenerations.

isometry class of type (U, −1) (nonsingular curve)						conjunction 1) conjunction 1') → Node (1)	conjunction 2) conjunction 2') → Node (2)	contraction 3) contraction 3') → Isolated point
r	a	δ _φ	g	k	g − 1	α, β	α, β	α, β
9	1	1	6	4	5	4, 4	4, 3	4, 4
9	1	1	6	4	5	5, 3	5, 2	5, 3
9	3	1	5	3	4	3, 3	3, 2	3, 3
9	3	1	5	3	4	4, 2	4, 1	4, 2
9	5	1	4	2	3	2, 2	2, 1	2, 2
9	5	1	4	2	3	3, 1	3, 0	3, 1
9	7	1	3	1	2	1, 1	1, 0	1, 1
9	7	1	3	1	2	2, 0	impossible	2, 0
9	9	1	2	0	1	0, 0	impossible	0, 0
9	9	1	2	0	1	impossible	impossible	impossible
9	9	1	2	0	1	conjunction 5) → Node (*)		
10	0	0	6	5	5	5, 4	5, 3	5, 4
10	0	0	6	5	5	5, 4	5, 3	5, 4
10	2	0	5	4	4	4, 3	4, 2	4, 3
10	2	0	5	4	4	4, 3	4, 2	4, 3
10	2	1	5	4	4	4, 3	4, 2	4, 3
10	2	1	5	4	4	4, 3	4, 2	4, 3
10	4	0	4	3	3	3, 2	3, 1	3, 2
10	4	0	4	3	3	3, 2	3, 1	3, 2
10	4	1	4	3	3	3, 2	3, 1	3, 2
10	4	1	4	3	3	3, 2	3, 1	3, 2
10	6	0	3	2	2	2, 1	2, 0	2, 1
10	6	0	3	2	2	2, 1	2, 0	2, 1
10	6	1	3	2	2	2, 1	2, 0	2, 1
10	6	1	3	2	2	2, 1	2, 0	2, 1
10	8	0	T ² ∪ T ²			conjunction 4) → Node (*)		
10	8	1	2	1	1	1, 0	impossible	1, 0
10	8	1	2	1	1	1, 0	impossible	1, 0
10	10	0	∅ (the empty real structure)					
10	10	1	1	0	0	impossible	impossible	impossible
10	10	1	1	0	0	impossible	impossible	impossible
11	1	1	5	5	4	5, 3	5, 2	5, 3
11	1	1	5	5	4	4, 4	4, 3	4, 4
11	3	1	4	4	3	4, 2	4, 1	4, 2
11	3	1	4	4	3	3, 3	3, 2	3, 3
11	5	1	3	3	2	3, 1	3, 0	3, 1
11	5	1	3	3	2	2, 2	2, 1	2, 2
11	7	1	2	2	1	2, 0	impossible	2, 0
11	7	1	2	2	1	1, 1	1, 0	1, 1
11	9	1	1	1	0	impossible	impossible	impossible
11	9	1	1	1	0	0, 0	impossible	0, 0
11	9	1	1	1	0	conjunction 5) → Node (*)		

TABLE 7. **Nonsingular** real curves in $|-2K_{\mathbb{F}_4}|$ on $\mathbb{R}\mathbb{F}_4$ and their possible degenerations. (continued)

isometry class of type $(\mathbb{U}, -1)$ (nonsingular curve)						conjunction 1) conjunction 1') → Node (1)	conjunction 2) conjunction 2') → Node (2)	contraction 3) contraction 3') → Isolated point
r	a	δ_φ	g	k	g - 1	α, β	α, β	α, β
12	2	1	4	5	3	5, 2	5, 1	5, 2
12	2	1	4	5	3	3, 4	3, 3	3, 4
12	4	1	3	4	2	4, 1	4, 0	4, 1
12	4	1	3	4	2	2, 3	2, 2	2, 3
12	6	1	2	3	1	3, 0	impossible	3, 0
12	6	1	2	3	1	1, 2	1, 1	1, 2
12	8	1	1	2	0	impossible	impossible	impossible
12	8	1	1	2	0	0, 1	0, 0	0, 1
13	3	1	3	5	2	5, 1	5, 0	5, 1
13	3	1	3	5	2	2, 4	2, 3	2, 4
13	5	1	2	4	1	4, 0	impossible	4, 0
13	5	1	2	4	1	1, 3	1, 2	1, 2
13	7	1	1	3	0	impossible	impossible	impossible
13	7	1	1	3	0	0, 2	0, 1	0, 2
14	2	0	3	6	2	6, 1	6, 0	6, 1
14	2	0	3	6	2	2, 5	2, 4	2, 5
14	4	0	2	5	1	5, 0	impossible	5, 0
14	4	0	2	5	1	1, 4	1, 3	1, 4
14	4	1	2	5	1	5, 0	impossible	5, 0
14	4	1	2	5	1	1, 4	1, 3	1, 4
14	6	1	1	4	0	impossible	impossible	impossible
14	6	1	1	4	0	0, 3	0, 2	0, 3
15	3	1	2	6	1	6, 0	impossible	6, 0
15	3	1	2	6	1	1, 5	1, 4	1, 5
15	5	1	1	5	0	impossible	impossible	impossible
15	5	1	1	5	0	0, 4	0, 3	0, 4
16	2	1	2	7	1	7, 0	impossible	7, 0
16	2	1	2	7	1	1, 6	1, 5	1, 6
16	4	1	1	6	0	impossible	impossible	impossible
16	4	1	1	6	0	0, 5	0, 4	0, 5
17	1	1	2	8	1	8, 0	impossible	8, 0
17	1	1	2	8	1	1, 7	1, 6	1, 7
17	3	1	1	7	0	impossible	impossible	impossible
17	3	1	1	7	0	0, 6	0, 5	0, 6
18	0	0	2	9	1	9, 0	impossible	9, 0
18	0	0	2	9	1	1, 8	1, 7	1, 8
18	2	0	1	8	0	impossible	impossible	impossible
18	2	0	1	8	0	0, 7	0, 6	0, 7
18	2	1	1	8	0	impossible	impossible	impossible
18	2	1	1	8	0	0, 7	0, 6	0, 7
19	1	1	1	9	0	impossible	impossible	impossible
19	1	1	1	9	0	0, 8	0, 7	0, 8

TABLE 8. **Nonsingular** real curves in $|-2K_{\mathbb{F}_4}|$ on $\mathbb{R}\mathbb{F}_4$ and their possible degenerations. (continued)

Now let us re-arrange the above data. We remove impossible cases.

- Theorem 15.**
- The degenerations conjunction 1) conjunction 2) contraction 3) of nonsingular curves are listed in **Table 9**,
 - the degenerations conjunction 1') conjunction 2') contraction 3') of nonsingular curves are listed in **Table 10**,
 - The degenerations conjunction 4) of nonsingular curves are listed in **Table 11**, and
 - The degenerations conjunction 5) of nonsingular curves are listed in **Table 12**. \square

isometry class of type $(\mathbb{U}, -1)$ (nonsingular curve)						conjunction 1)	conjunction 2)	contraction 3)	No.
r	a	δ_φ	g	k	g - 1	\rightarrow Node(1)	\rightarrow Node(2)	\rightarrow Isolated point	
						α, β	α, β	α, β	
1	1	1	10	0	9	0, 8	0, 7	0, 8	1
2	0	0	10	1	9	1, 8	1, 7	1, 8	2
2	2	0	9	0	8	0, 7	0, 6	0, 7	3
2	2	1	9	0	8	0, 7	0, 6	0, 7	4
3	1	1	9	1	8	1, 7	1, 6	1, 7	5
3	3	1	8	0	7	0, 6	0, 5	0, 6	6
4	2	1	8	1	7	1, 6	1, 5	1, 6	7
4	4	1	7	0	6	0, 5	0, 4	0, 5	8
5	3	1	7	1	6	1, 5	1, 4	1, 5	9
5	5	1	6	0	5	0, 4	0, 3	0, 4	10
6	2	0	7	2	6	2, 5	2, 4	2, 5	11
6	4	0	6	1	5	1, 4	1, 3	1, 4	12
6	4	1	6	1	5	1, 4	1, 3	1, 4	13
6	6	1	5	0	4	0, 3	0, 2	0, 3	14
7	3	1	6	2	5	2, 4	2, 3	2, 4	15
7	5	1	5	1	4	1, 3	1, 2	1, 3	16
7	7	1	4	0	3	0, 2	0, 1	0, 2	17
8	2	1	6	3	5	3, 4	3, 3	3, 4	18
8	4	1	5	2	4	2, 3	2, 2	2, 3	19
8	6	1	4	1	3	1, 2	1, 1	1, 2	20
8	8	1	3	0	2	0, 1	0, 0	0, 1	21
9	1	1	6	4	5	4, 4	4, 3	4, 4	22
9	3	1	5	3	4	3, 3	3, 2	3, 3	23
9	5	1	4	2	3	2, 2	2, 1	2, 2	24
9	7	1	3	1	2	1, 1	1, 0	1, 1	25
9	9	1	2	0	1	0, 0	impossible	0, 0	26
10	0	0	6	5	5	5, 4	5, 3	5, 4	27
10	2	0	5	4	4	4, 3	4, 2	4, 3	28
10	2	1	5	4	4	4, 3	4, 2	4, 3	29
10	4	0	4	3	3	3, 2	3, 1	3, 2	30
10	4	1	4	3	3	3, 2	3, 1	3, 2	31
10	6	0	3	2	2	2, 1	2, 0	2, 1	32
10	6	1	3	2	2	2, 1	2, 0	2, 1	33
10	8	1	2	1	1	1, 0	impossible	1, 0	34
11	1	1	5	5	4	5, 3	5, 2	5, 3	35
11	3	1	4	4	3	4, 2	4, 1	4, 2	36
11	5	1	3	3	2	3, 1	3, 0	3, 1	37
11	7	1	2	2	1	2, 0	impossible	2, 0	38
12	2	1	4	5	3	5, 2	5, 1	5, 2	39
12	4	1	3	4	2	4, 1	4, 0	4, 1	40
12	6	1	2	3	1	3, 0	impossible	3, 0	41
13	3	1	3	5	2	5, 1	5, 0	5, 1	42
13	5	1	2	4	1	4, 0	impossible	4, 0	43
14	2	0	3	6	2	6, 1	6, 0	6, 1	44
14	4	0	2	5	1	5, 0	impossible	5, 0	45
14	4	1	2	5	1	5, 0	impossible	5, 0	46
15	3	1	2	6	1	6, 0	impossible	6, 0	47
16	2	1	2	7	1	7, 0	impossible	7, 0	48
17	1	1	2	8	1	8, 0	impossible	8, 0	49
18	0	0	2	9	1	9, 0	impossible	9, 0	50

TABLE 9. \mathbb{F}_4 : Conjunction 1), Conjunction 2) and Contraction 3)

isometry class of type $(\mathbb{U}, -1)$ (nonsingular curve)						conjunction 1')	conjunction 2')	contraction 3')	
r	a	δ_φ	g	k	g - 1	\rightarrow Node(1)	\rightarrow Node(2)	\rightarrow Isolated point	No.
19	1	1	1	9	0	α, β	α, β	α, β	1'
18	0	0	2	9	1	1, 8	1, 7	1, 8	2'
18	2	0	1	8	0	0, 7	0, 6	0, 7	3'
18	2	1	1	8	0	0, 7	0, 6	0, 7	4'
17	1	1	2	8	1	1, 7	1, 6	1, 7	5'
17	3	1	1	7	0	0, 6	0, 5	0, 6	6'
16	2	1	2	7	1	1, 6	1, 5	1, 6	7'
16	4	1	1	6	0	0, 5	0, 4	0, 5	8'
15	3	1	2	6	1	1, 5	1, 4	1, 5	9'
15	5	1	1	5	0	0, 4	0, 3	0, 4	10'
14	2	0	3	6	2	2, 5	2, 4	2, 5	11'
14	4	0	2	5	1	1, 4	1, 3	1, 4	12'
14	4	1	2	5	1	1, 4	1, 3	1, 4	13'
14	6	1	1	4	0	0, 3	0, 2	0, 3	14'
13	3	1	3	5	2	2, 4	2, 3	2, 4	15'
13	5	1	2	4	1	1, 3	1, 2	1, 2	16'
13	7	1	1	3	0	0, 2	0, 1	0, 2	17'
12	2	1	4	5	3	3, 4	3, 3	3, 4	18'
12	4	1	3	4	2	2, 3	2, 2	2, 3	19'
12	6	1	2	3	1	1, 2	1, 1	1, 2	20'
12	8	1	1	2	0	0, 1	0, 0	0, 1	21'
11	1	1	5	5	4	4, 4	4, 3	4, 4	22'
11	3	1	4	4	3	3, 3	3, 2	3, 3	23'
11	5	1	3	3	2	2, 2	2, 1	2, 2	24'
11	7	1	2	2	1	1, 1	1, 0	1, 1	25'
11	9	1	1	1	0	0, 0	impossible	0, 0	26'
10	0	0	6	5	5	5, 4	5, 3	5, 4	27'
10	2	0	5	4	4	4, 3	4, 2	4, 3	28'
10	2	1	5	4	4	4, 3	4, 2	4, 3	29'
10	4	0	4	3	3	3, 2	3, 1	3, 2	30'
10	4	1	4	3	3	3, 2	3, 1	3, 2	31'
10	6	0	3	2	2	2, 1	2, 0	2, 1	32'
10	6	1	3	2	2	2, 1	2, 0	2, 1	33'
10	8	1	2	1	1	1, 0	impossible	1, 0	34'
9	1	1	6	4	5	5, 3	5, 2	5, 3	35'
9	3	1	5	3	4	4, 2	4, 1	4, 2	36'
9	5	1	4	2	3	3, 1	3, 0	3, 1	37'
9	7	1	3	1	2	2, 0	impossible	2, 0	38'
8	2	1	6	3	5	5, 2	5, 1	5, 2	39'
8	4	1	5	2	4	4, 1	4, 0	4, 1	40'
8	6	1	4	1	3	3, 0	impossible	3, 0	41'
7	3	1	6	2	5	5, 1	5, 0	5, 1	42'
7	5	1	5	1	4	4, 0	impossible	4, 0	43'
6	2	0	7	2	6	6, 1	6, 0	6, 1	44'
6	4	0	6	1	5	5, 0	impossible	5, 0	45'
6	4	1	6	1	5	5, 0	impossible	5, 0	46'
5	3	1	7	1	6	6, 0	impossible	6, 0	47'
4	2	1	8	1	7	7, 0	impossible	7, 0	48'
3	1	1	9	1	8	8, 0	impossible	8, 0	49'
2	0	0	10	1	9	9, 0	impossible	9, 0	50'

TABLE 10. \mathbb{F}_4 : Conjunction 1'), Conjunction 2') and Contraction 3')

isometry class of type $(\mathbb{U}, -1)$ (nonsingular curve)						conjunction 4)
r	a	δ_φ	g	k	g - 1	
10	8	0	$T^2 \cup T^2$			Node (*)
10	8	0	$T^2 \cup T^2$			Node (*)

TABLE 11. \mathbb{F}_4 : Conjunction 4)

isometry class of type $(\mathbb{U}, -1)$ (nonsingular curve)						conjunction 5)
r	a	δ_φ	g	k	g-1	
9	9	1	2	0	1	Node (*)
11	9	1	1	1	0	Node (*)

TABLE 12. \mathbb{F}_4 : Conjunction 5)

We can verify the following lemma, which is an analogy of Lemma 11, by checking the data in the Tables 6 — 8 and Tables 4 — 5 above.

Lemma 16. *Fix an isometry class of involutions of \mathbb{L}_{K3} of type $(\mathbb{U}, -1)$ with $(r, a, \delta_{\varphi S}, v) \neq (10, 8, 0, 0), (10, 10, 0, 0)$, and find a corresponding real 2-elementary K3 surface (X, φ) of type $(\mathbb{U}, -1)$ and a real **nonsingular** curve A_1 in $|12c + 3s|$ on $\mathbb{R}\mathbb{F}_4$.*

- For real curves A'_1 with **one nondegenerate double point** on \mathbb{F}_4 being the results of degenerations of types 1)—3) of the real **nonsingular** curve A_1 above, we should **choose** “ φ_- ” (Definition 21 and Lemma 18) as the anti-holomorphic involutions on X . And for all marked real 2-elementary K3 surfaces $((X, \tau, \varphi_-), \alpha)$ obtained from A'_1 , the associated involutions $\alpha \circ (\varphi_-)_* \circ \alpha^{-1}$ of \mathbb{L}_{K3} are **isometric** with respect to $G = \{1_S\}$, namely, all such (X, τ, φ_-) have their markings to the same involution of \mathbb{L}_{K3} .
- For real curves A'_1 with **one nondegenerate double point** on \mathbb{F}_4 being the results of degenerations of types 1')—3') of the real **nonsingular** curve A_1 above, we should **choose** “ φ_- ” (Definition 21 and Lemma 18) as the anti-holomorphic involutions on X . And for all marked real 2-elementary K3 surfaces $((X, \tau, \varphi_-), \alpha)$ obtained from A'_1 , the associated involutions $\alpha \circ (\varphi_-)_* \circ \alpha^{-1}$ of \mathbb{L}_{K3} are **isometric** with respect to $G = \{1_S\}$, namely, all such (X, τ, φ_-) have their markings to the same involution of \mathbb{L}_{K3} . \square

8.3. Conjectures. We set $(S, \theta) := ((3, 1, 1), -1)$. Note $G = \{1_S\}$ for $(S, \theta) := ((3, 1, 1), -1)$. See Remark 5. As stated in Section 5, we have an orthogonal decomposition

$$S \cong \mathbb{U} \oplus \mathbb{Z}(F),$$

where \mathbb{U} is the hyperbolic even unimodular lattice of signature $(1, 1)$ and $\mathbb{Z}(F) \cong \langle -2 \rangle$.

We **fix** an involution (\mathbb{L}_{K3}, ψ) of type $((3, 1, 1), -1)$ and use the notation

$$\Omega / -\psi = \mathcal{L}_+ \times \mathcal{L}_{-,S},$$

e.t.c. defined in Section 2.3.

We want to give a **criterion** for a real curve $\mathbb{R}A'_1$ in $|12c + 3s|$ on \mathbb{F}_4 with one double point to be **nondegenerate** in terms of the involution (\mathbb{L}_{K3}, ψ) .

Conjecture 1 (A criterion, cf. Lemma 7). *For $[\omega] \in \Omega / -\psi$, $[\omega]$ is the period of a marked real 2-elementary K3 surface obtained from a real curve $\mathbb{R}A'_1$ in $|12c + 3s|$ with one **nondegenerate** double point on \mathbb{F}_4*
 \iff *there are no $\mathbf{v} (\neq \pm F)$ in \mathbb{L}_{K3} satisfying that $\mathbf{v} \cdot \omega = 0$, $\mathbf{v} \cdot \mathbb{U} = 0$ and $\mathbf{v}^2 = -2$. \square*

At present we can prove the direction \Leftarrow of the above assertion.

However, we can prove the following lemmatae.

Lemma 17. *Either \mathbb{L}_+ or $\mathbb{L}_{-,S}$ does not contain any element \mathbf{v} such that $\mathbf{v} \equiv F \pmod{2\mathbb{L}_{K3}}$. Hence, we have the following. If $\mathbb{L}_{-,S}$ contains an element \mathbf{v} such that $\mathbf{v} \equiv F \pmod{2\mathbb{L}_{K3}}$, then \mathbb{L}_+ does not contain any \mathbf{v} such that $\mathbf{v} \equiv F \pmod{2\mathbb{L}_{K3}}$. \square*

Lemma 18. *For the anti-holomorphic involution “ φ_- ” (recall its definition) $\mathbb{L}_{-,S}$ contains an element \mathbf{v} such that $\mathbf{v} \equiv F \pmod{2\mathbb{L}_{K3}}$. \square*

(-2) -orthogonal hyperplanes and their reflections. The reflection on \mathcal{L}_+ with respect to the real hyperplane \mathbf{v}^\perp where \mathbf{v} is an element of \mathbb{L}_+ with square -2 is welldefined and it sends a point in $\Omega / -\psi$ to its equivalent point. Also, the reflection on $\mathcal{L}_{-,S}$ with respect to the real hyperplane \mathbf{v}^\perp where \mathbf{v} is an element of $\mathbb{L}_{-,S}$ with square -2 is welldefined and it sends a point in $\Omega / -\psi$ to its equivalent point.

Hence, let

$$\Omega_+ \text{ (respectively, } \Omega_{-,S})$$

be **the open fundamental domains** with respect to the groups generated by the reflections with respect to the real hyperplanes \mathbf{v}^\perp satisfying $\mathbf{v}^2 = -2$ and $\mathbf{v} \in \mathbb{L}_+$ (respectively, $\mathbb{L}_{-,S}$) and we consider the direct product

$$\Omega_+ \times \Omega_{-,S}.$$

We want to obtain an analogy of Theorem 9. We have to formulate some domain $\Omega_*^{\mathbb{R}F_4}$ like the domain $\Omega_*^{\mathbb{R}P^2}$.

Conjecture 2 (cf. [3], Theorem 2.1). *We fix an involution (\mathbb{L}_{K3}, ψ) of type $((3, 1, 1), -1)$. Rigid isotopy classes (up to projective equivalence) of real curves $\mathbb{R}A'_1$ in $|12c + 3s|$ on \mathbb{F}_4 with one non-degenerate double point which yield marked real 2-elementary K3 surfaces $((X, \tau, \varphi), \alpha)$ satisfying $\alpha\varphi_*\alpha^{-1} = \psi$ are in one to one (bijective) correspondence with the connected components (“polytopes”) (considered up to equivalence) of “some domain” $\Omega_*^{\mathbb{R}F_4}$. \square*

The Coxeter graph C and the graph K . We are fixing an involution (\mathbb{L}_{K3}, ψ) of type $((3, 1, 1), -1)$. We set

$$\mathbb{L}_{-,U} := \mathbb{L}_- \cap U^\perp \cong \mathbb{L}_- \cap (U \oplus U \oplus (-E_8) \oplus (-E_8)).$$

Here recall that $\mathbb{L}_\pm := \{x \in \mathbb{L}_{K3} \mid \psi(x) = \pm x\}$.

Then $\mathbb{L}_{-,U}$ is also hyperbolic. Let

$$\mathcal{L}_{-,U}$$

be the Lobachevsky space obtained from $\mathbb{L}_{-,U} \otimes \mathbb{R}$.

The group generated by the reflections with respect to real hyperplanes \mathbf{v}^\perp such that $\mathbf{v} \in \mathbb{L}_{-,U}$ with $\mathbf{v}^2 = -2$ acts on the Lovachevskii space $\mathcal{L}_{-,U}$. Let

$$\widetilde{\Omega}_-$$

be one of its fundamental domains (polytope) which has a face orthogonal to F .

Let

$$C$$

be the Coxeter graph (see Vinberg [11]) of $\widetilde{\Omega}_-$.

Definition 27. • *Let C' be the graph which is obtained by removing all thick or dotted edges from C .*

- *Consider the group of symmetries of C' obtained from some automorphism of $(\mathbb{L}_{K3}, \psi, U)$. Let C'' be the quotient graph of C' by the action of the group.*
- *Let e be the vertex of C corresponding to F and let e' be the class (in C'') containing e .*
- *Let K be the connected component of C'' containing e' . \square*

Conjecture 3 (cf. [3], Proposition 3.1). *The number (up to equivalence) of polytopes of $\Omega_*^{\mathbb{R}F_4}$ (in Conjecture 2) coincides with that of vertices of the graph K . \square*

The graph P . We fix an isometry class of integral involutions of type $(U, -1)$ with $(r, a, \delta_{\varphi S}, v) \neq (10, 8, 0, 0), (10, 10, 0, 0)$. Then we have a real 2-elementary K3 surface (X, τ, φ) of type $(U, -1)$ and a real **nonsingular** curve A_1 in $|12c + 3s|$ on $\mathbb{R}F_4$. (See Definition 26.)

Then, we define the graphs P as follows.

- Definition 28.** • The vertices of P are all the **rigid isotopy classes** of real curves A'_1 in $|12c + 3s|$ with one nondegenerate double point on \mathbb{F}_4 being the results of degenerations of types 1)—3) of the real nonsingular curve A_1 with φ above.
- Two vertices of P are connected by a **edge** if one rigid isotopy class is obtained from the conjunction of the ovals \mathcal{E}_1 and the nontrivial component of A_1 (Conjunction 1)), and the other rigid isotopy class is obtained from the contraction of the oval \mathcal{E}_1 (Contraction 3)).
 - Two vertices of P are connected by a **edge** if one rigid isotopy class is obtained from the conjunction of the ovals \mathcal{E}_1 and \mathcal{E}_2 of A_1 (Conjunction 2)), and the other rigid isotopy class is obtained from the contraction of the oval \mathcal{E}_1 (Contraction 3)). \square

The relation of K and P .

Conjecture 4 (cf. [3], Proposition 3.4). Fix a real **nonsingular** curve A_1 in $|12c + 3s|$ on \mathbb{F}_4 with φ as above.

Let us get a arbitrary real curve A'_1 in $|12c + 3s|$ on \mathbb{F}_4 with one nondegenerate double point being the result of one of degenerations of types 1)—3) of the nonsingular curve A_1 with φ , and construct the graph K (Definition 27) from A'_1 and φ_- (Recall Theorem 12). Then, K is isomorphic to P . \square

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